Lecture 8: Logit and Probit Models

- 8.1 Logit Models
  - 8.1.1 Example: SP Survey in the Audience
- 8.2 Probit Models
- 8.3 Elasticities
  - 8.3.1 Microscopic
  - 8.3.2 Macroscopic

8.1 Logit Models: Definition

All Logit models are defined by **Gumbel-distributed** random utilities.

- The standard **Multinomial-Logit model (MNL)** has RUs distributed according to $\epsilon_i \sim$ i.i.d Gumbel$(0, 1)$
- Distribution:

$$F_{\text{Gu}}^{(\eta,\lambda)}(x) = \exp\left[-e^{-\lambda(x-\eta)}\right]$$

- Density:

$$f_{\text{Gu}}^{(\eta,\lambda)}(x) = \frac{\mathrm{d}F_{\text{Gu}}^{(\eta,\lambda)}(x)}{\mathrm{d}x} = \lambda e^{-\lambda(x-\eta)} \exp\left[-e^{-\lambda(x-\eta)}\right].$$

- Statistical properties:

$$\epsilon_{\text{mode}} = \eta, \quad E(\epsilon) = \eta + \gamma/\lambda \text{ with } \gamma = 0.5772, \quad V(\epsilon) = \frac{\pi^2}{6\lambda^2}$$

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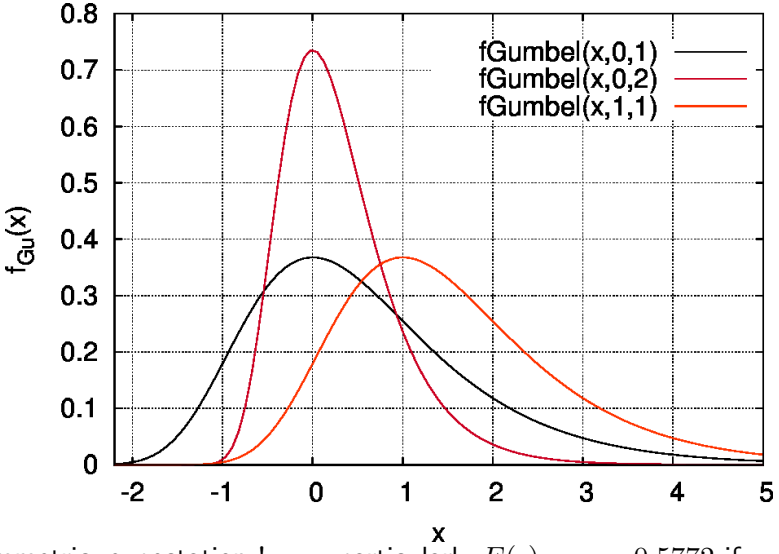
Density functions of some Gumbel distributions



$\Rightarrow$ not symmetric; expectation $! = \eta$, particularly $E(\epsilon) = \gamma = 0.5772$ if $\epsilon \sim \text{Gu}(0, 1)$

Questions

? The numerical values of the deterministic utilities V_i are $\pi/\sqrt{6} \approx 1.28$ times as large as if the RU variance $V(\epsilon)$ were $= 1$. Why?

Because of the **scaling invariance** of discrete-choice models: The choice probability remains unchanged if *both* the random and deterministic utilities are multiplied by a factor $\lambda > 0$

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This is due to the **translation invariance** of discrete-choice models: When adding a real-valued constant to the utilities of all alternatives, nothing changes. Here, a common $E(\epsilon_i) = 0.5772$ (remember, $\epsilon \sim i.i.d.!$) is just such a common constant.

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Gumbel distribution as a limit distribution for max(.)

The maximum of many i.i.d. random variables X_i with exponential tails $\propto \exp(-\lambda x)$ approaches a Gumbel or Generalized Extreme Value Type-I distribution:

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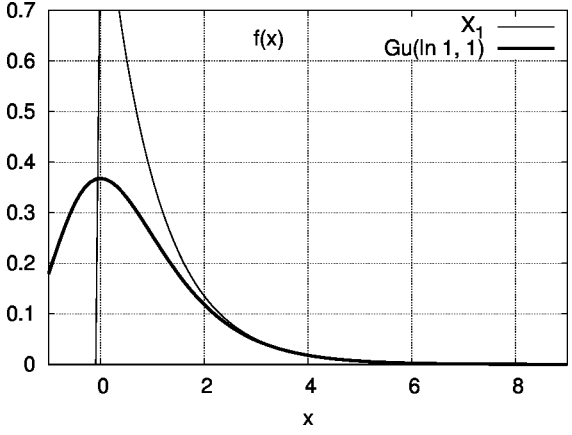

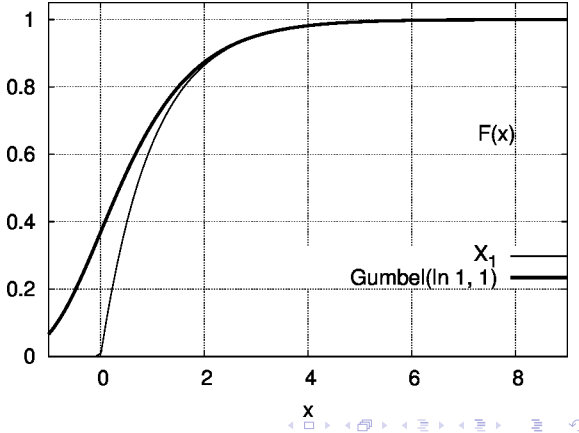

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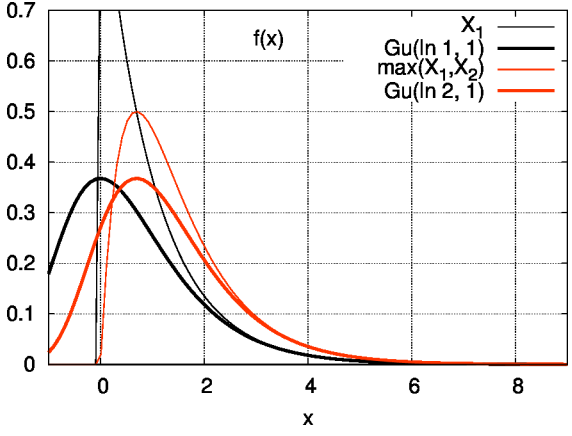

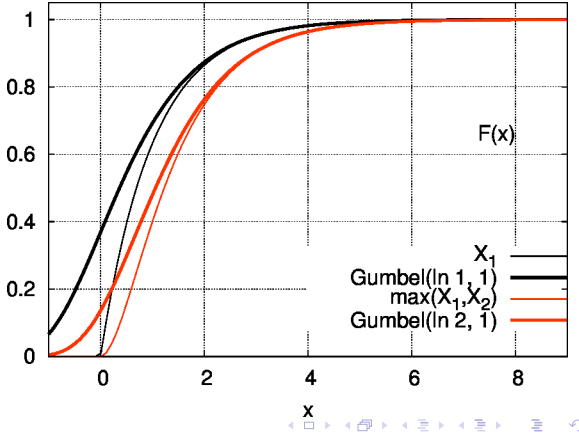

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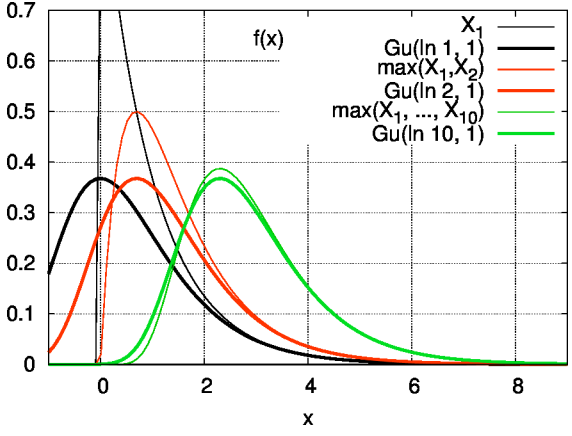

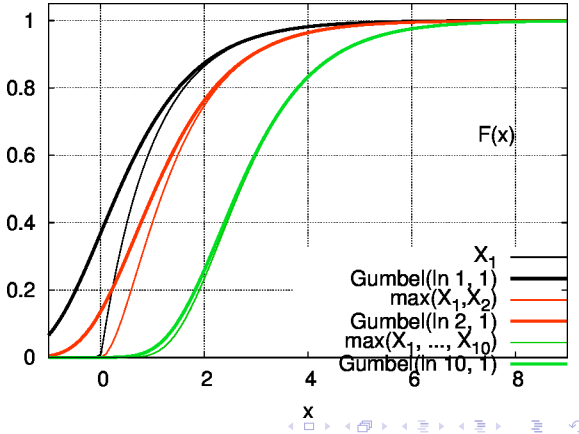

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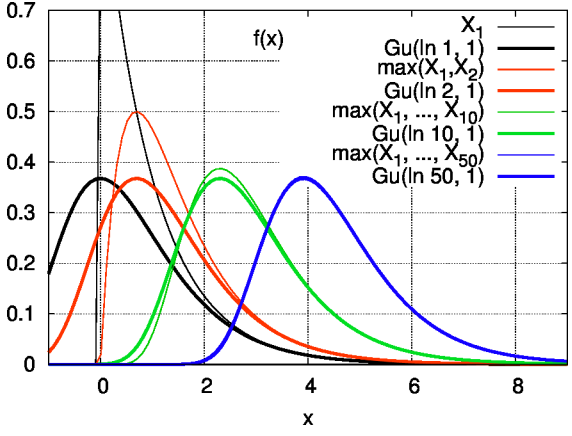

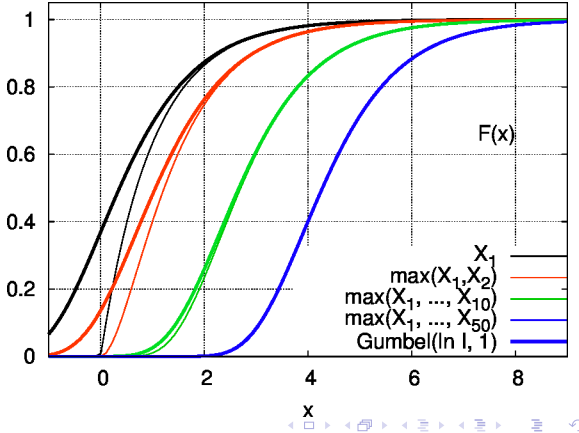

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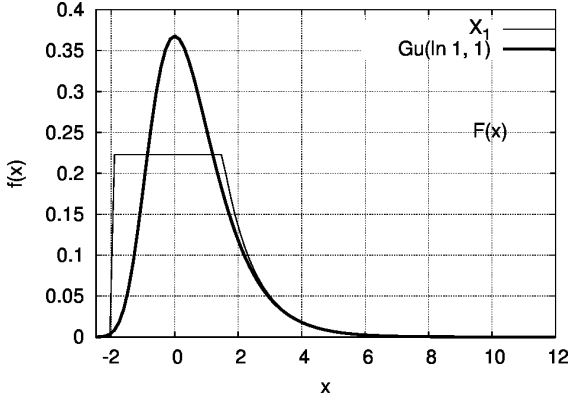

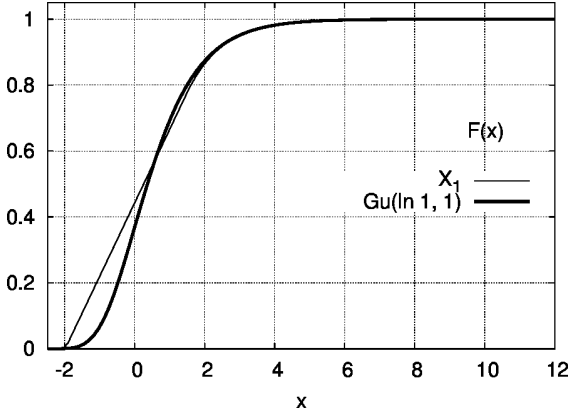

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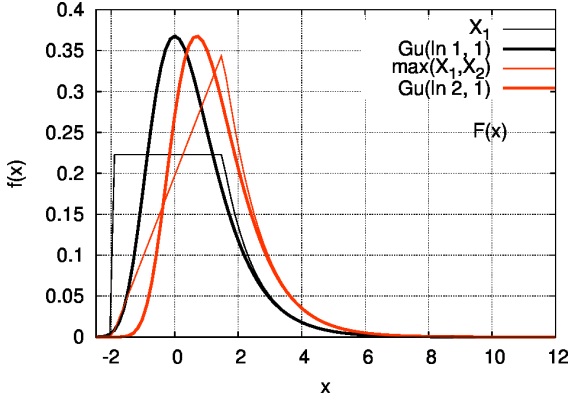

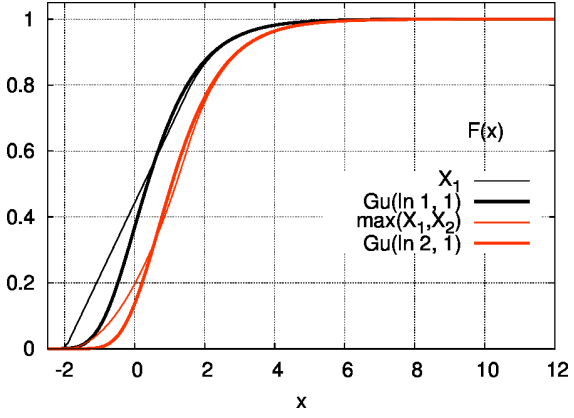

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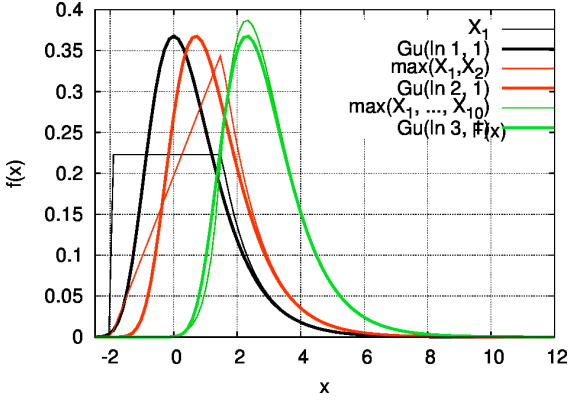

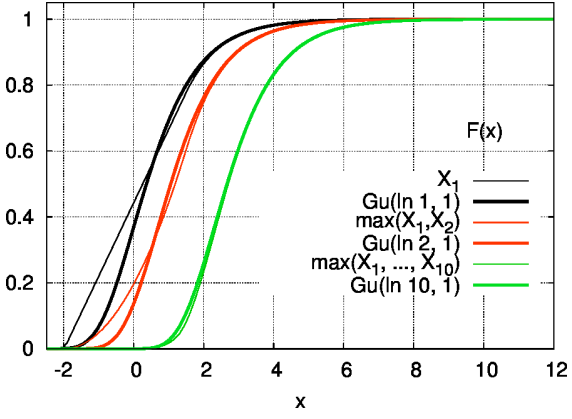

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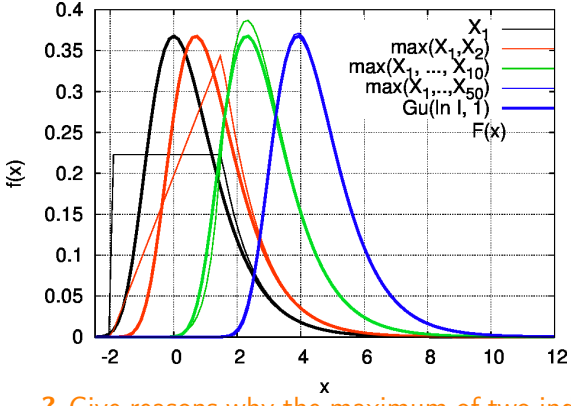

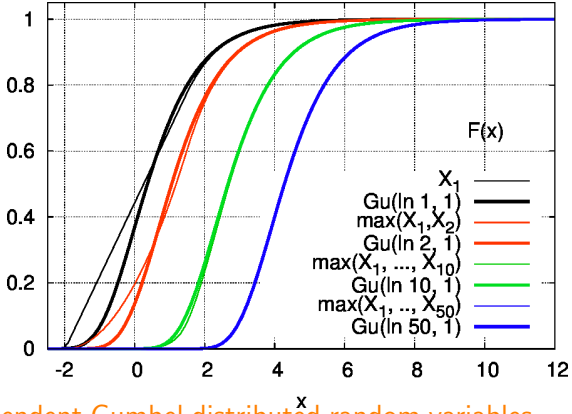

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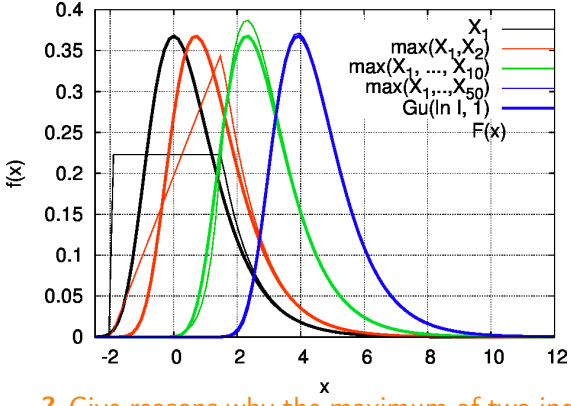

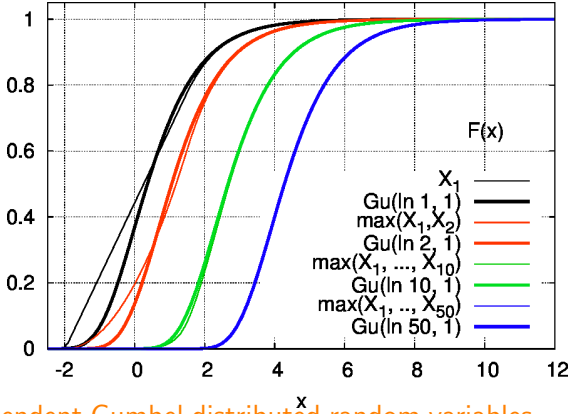

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Since $\max(\max(x_1, x_2), \max(x_3, x_4)) = \max(x_1, x_2, x_3, x_4)$

Properties of the Multinomial-Logit Model (MNL)

- Models of the Logit family (MNL, nested Logit, GEV models) are the only ones with explicit expressions for the choice probabilities for the multinomial case $I > 2$. For the MNL itself, we have

$$P_i^{\mathsf{MNL}} = \frac{\exp(V_i)}{\sum_j \exp(V_j)}$$

- Besides the translational and scale invariance of all simple discrete-choice models, the MNL has the **Independence of Irrelevant Alternatives (IIA)** property:

> IIA property: The relative preference of Alternative i over j as defined by the choice probability ratio P_i/P_j does not depend on other alternatives $k \neq i, j$

- The IIA property is exclusively true for the MNL. In fact, the MNL can be equivalently defined by the IIA property instead of i.i.d. Gumbel RUs.

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Questions (2)

? The choice probabilities of three alternatives are given by $P_1 = 0.2$, $P_2 = 0.4$, and $P_3 = 0.4$. Now, Alternative 3 is no longer available. Give the new Logit choice probabilities.

No need to re-calculate using the Logit probability formula. Just use the IIA property:

$$\frac{P_1}{P_2} = 1/2 = \text{const.} \;\Rightarrow\; P_1 = 1/3,\; P_2 = 2/3$$

? The Gumbel distribution is the limit distribution of the maximum of exponentially-tailed random variables. Is there really a justification for this sort of distribution if the RUs are the result of many unknown/not considered effects?

Not really. If there are many unknown/not considered effects, the chance is high that they are not correlated and the **central limit theorem** can be applied (even if there are correlations, this theorem is quite robust). Hence, there would be a justification for Gaussian rather than Gumbel RUs. The fact that the maximum of exponentially-tailed distributions is Gumbel distributed has no real relevance here.

Questions (2)

? The choice probabilities of three alternatives are given by $P_1 = 0.2$, $P_2 = 0.4$, and $P_3 = 0.4$. Now, Alternative 3 is no longer available. Give the new Logit choice probabilities.

No need to re-calculate using the Logit probability formula. Just use the IIA property:

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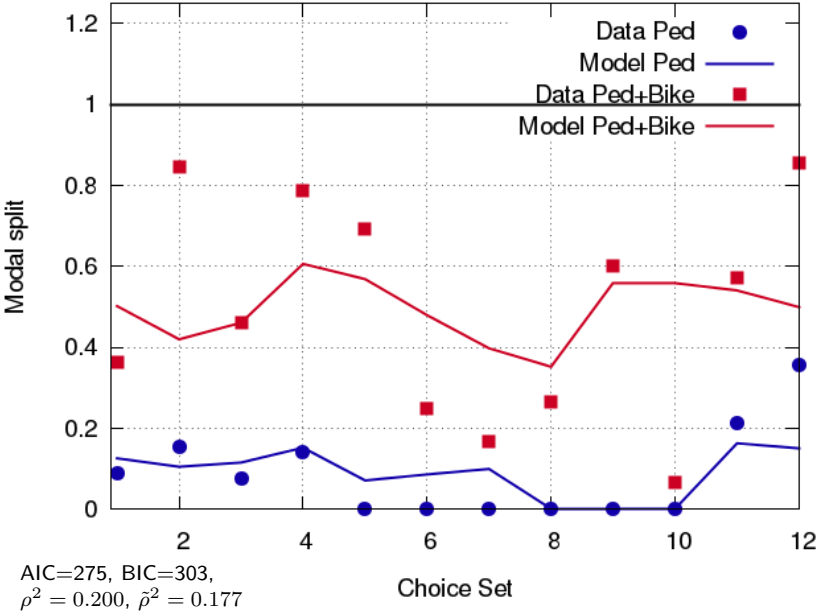
8.1.1 Example: SP Survey in the audience WS18/19 (red: bad weather, $W = 1$)

Choice Set	Alt. 1: Ped	Alt. 2: Bike	Alt. 3: PT/Car	Alt 1	Alt 2	Alt 3
1	30 min	20 min	20 min+0€	1	3	7
2	30 min	20 min	20 min+2€	2	9	2
3	30 min	20 min	20 min+1€	1	5	7
4	30 min	20 min	30 min+0€	2	9	3
5	50 min	20 min	30 min+0€	0	9	4
6	50 min	30 min	30 min+0€	0	3	9
7	50 min	40 min	30 min+0€	0	2	10
8	180 min	60 min	60 min+2€	0	4	11
9	180 min	40 min	60 min+2€	0	9	6
10	180 min	40 min	60 min+2€	0	1	14
11	12 min	8 min	10 min+0€	3	5	6
12	12 min	8 min	10 min+1€	5	7	2

Model 1: generic times and costs, no weather



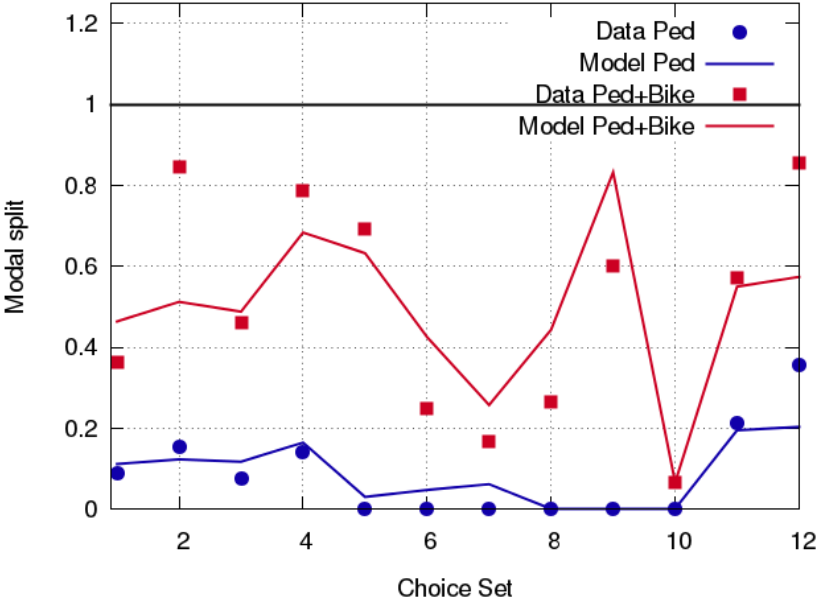
$$V_i = \beta_0\delta_{i1} + \beta_1\delta_{i2} + \beta_2 K_i + \beta_3 T_i$$

$$\begin{aligned}\beta_0 &= -0.95 \pm 0.37,\\ \beta_1 &= -0.28 \pm 0.24,\\ \beta_2 &= +0.17 \pm 0.19,\\ \beta_3 &= -0.04 \pm 0.02\end{aligned}$$

$$\frac{\beta_0}{-\beta_3} = -22.4\,\text{min},$$
$$\frac{\beta_1}{-\beta_3} = -6.6\,\text{min},$$
$$\frac{60\beta_3}{\beta_2} = -15\,€/\text{h}$$

AIC=275, BIC=303,
$\rho^2 = 0.200$, $\tilde{\rho}^2 = 0.177$

Model 2: generic times/costs plus weather factor (bad weather, $W = 1$)

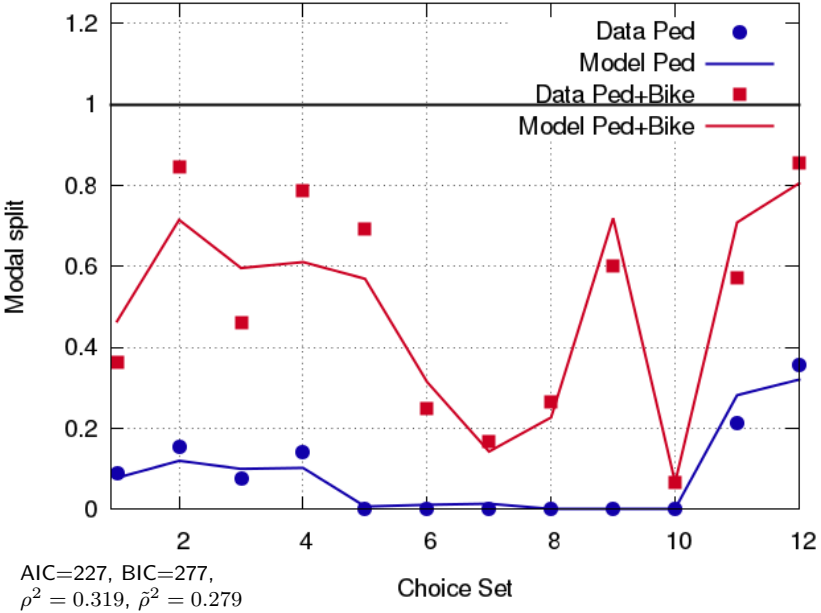


$$
\begin{aligned}
V_i &= \beta_0 \delta_{i1} + \beta_1 \delta_{i2} \\
&+ \beta_2 K_i + \beta_3 T_i \\
&+ \beta_4 W \delta_{i3}
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 &= -0.65 \pm 0.37, \\
\beta_1 &= -0.42 \pm 0.25, \\
\beta_2 &= -0.10 \pm 0.20, \\
\beta_3 &= -0.09 \pm 0.02, \\
\beta_4 &= 4.2 \pm 1.1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\beta_0}{-\beta_3} &= -7.1\,\text{min}, \\
\frac{\beta_1}{-\beta_3} &= -4.6\,\text{min}, \\
\frac{\beta_0}{-\beta_2} &= -6.7\,€, \\
\frac{\beta_1}{-\beta_2} &= -4.3\,€, \\
\frac{60\beta_3}{\beta_2} &= +57\,€/\text{h}, \\
\frac{\beta_4}{-\beta_2} &= +44\,€
\end{aligned}
$$

Model 3: alt-spec time sensitivities plus weather factor



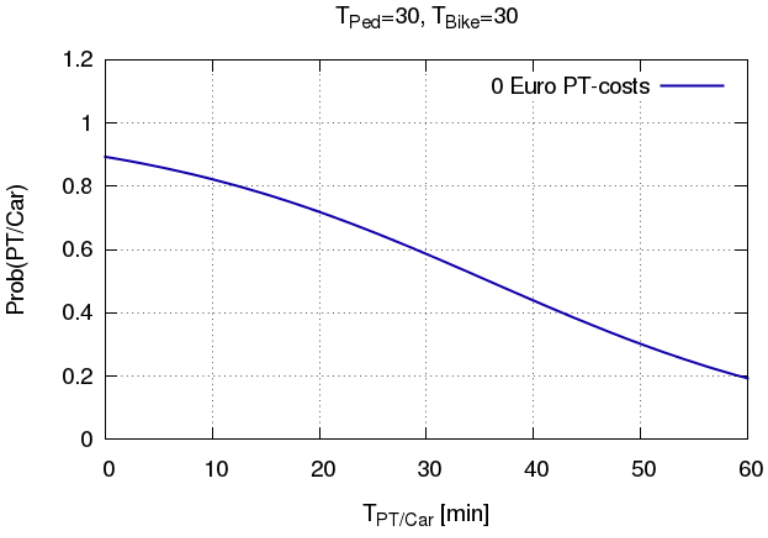
AIC=227, BIC=277,
$\rho^2 = 0.319$, $\tilde{\rho}^2 = 0.279$

$$
\begin{aligned}
V_i &= \beta_0 \delta_{i1} + \beta_1 \delta_{i2} \\
&+ \beta_2 K + \beta_3 T_1 \delta_{i1} \\
&+ \beta_4 T_2 \delta_{i2} + \beta_5 T_3 \delta_{i3} \\
&+ \beta_6 W \delta_{i3}
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 &= +1.03 \pm 0.74, \\
\beta_1 &= +0.66 \pm 0.40, \\
\beta_2 &= -0.53 \pm 0.25, \\
\beta_3 &= -0.14 \pm 0.03, \\
\beta_4 &= -0.11 \pm 0.03, \\
\beta_5 &= -0.06 \pm 0.03, \\
\beta_6 &= +3.6 \pm 1.1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\beta_0}{-\beta_3} &= +7.5\,\text{min}, \\
\frac{\beta_1}{-\beta_3} &= +4.7\,\text{min}, \\
\frac{\beta_0}{-\beta_2} &= +1.9\,€, \\
\frac{\beta_1}{-\beta_2} &= +1.2\,€, \\
\frac{60\beta_5}{\beta_2} &= +6.7\,€/\text{h}, \\
\frac{60\beta_4}{-\beta_2} &= +12.5\,€/\text{h}
\end{aligned}
$$

Model 3: Dependence of the modal split on the PT attributes



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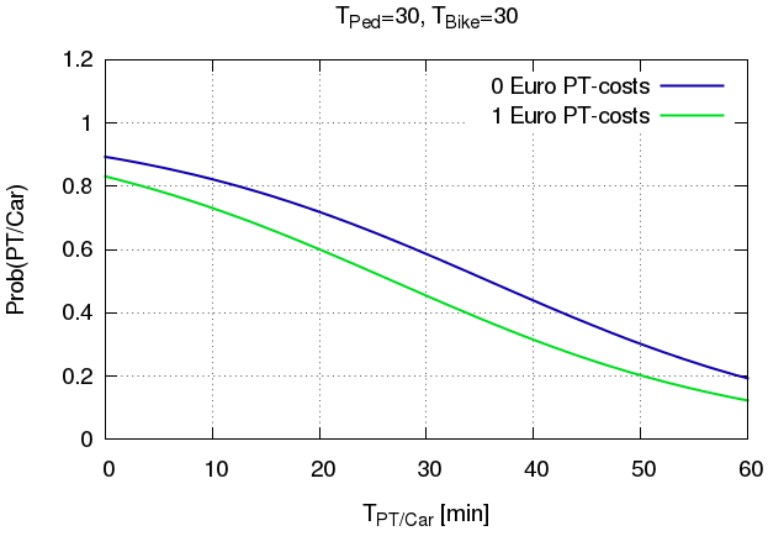

Everything plausible

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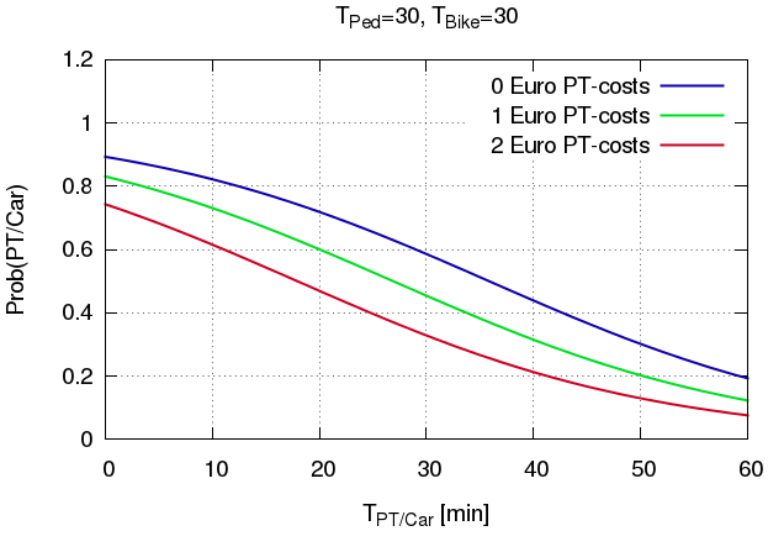

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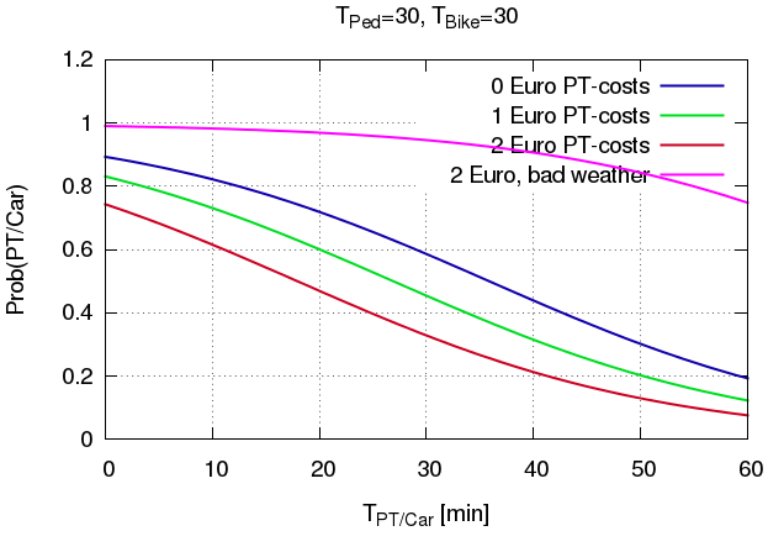
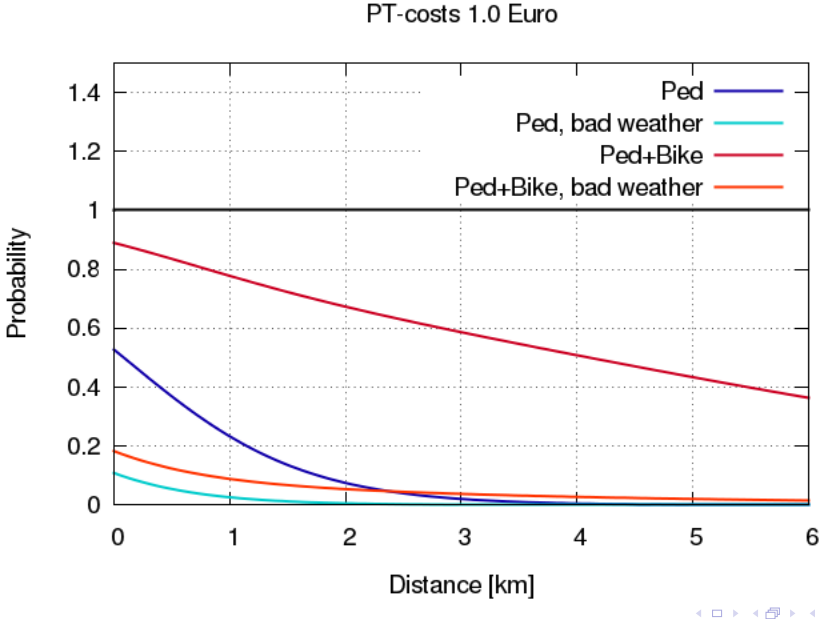


Everything plausible

Model 3: Dependence on the distance

assuming plausible speeds 5, 15, and 25 km/h for each mode, respectively



8.2 Probit Models

The **Probit Model** class is defined by (generally correlated) Gaussian RUs.

- The general multinomial Probit model (MNP) has random utilities $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ with the variance-covariance matrix $\boldsymbol{\Sigma}$ of the RUs
- The special case of the i.i.d. MNP with $\boldsymbol{\Sigma} = \mathbf{1}$ (unit matrix), i.e., $\epsilon_i \sim$ i.i.d. $N(0, 1)$ has similar properties as the MNL (but not the IIA property!). However, for $I \geq 3$, the MNP needs integrals (1d, if there are no correlations) to calculate the choice probabilities.
- Often, i.i.d Gaussian RUs can be motivited by the central-limit theorem while Gumbel distributed ones cannot. However, since the MNL behaves similarly and has explicit choice probabilities and a simpler calibration, it is often favoured over the i.i.d. MNP.

? Why one can set the variance-covariance matrix to be the unit matrix (i.e. setting all variances=1) in case of the i.i.d MNP?

Because of the Scaling invariance of all Discrete-choice models with additive random utilities. If we had $\epsilon_i \sim$ i.i.d. $N(0, 1/\lambda^2)$, just multiply the deterministic and random utilities by λ to have an equivalent Probit model with $\epsilon_i \sim$ i.i.d. $N(0, 1)$.

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Choice probabilities of the binary Probit model I

- Choice probabilities of the binary Probit model with $\epsilon_i \sim$ i.i.d. $N(0,1)$:

$$P_1 = \Phi\left(\frac{V_1 - V_2}{\sqrt{2}}\right), \quad P_2 = 1 - P_1$$

? Derive the choice probabilities for the correlated binary Probit model. *Hint:* a linear combination of Gaussians is again a Gaussian

Assume without loss of generality zero expectations and use the general rules for the variance of two random variables X_1, X_2 ($a, b \in \mathbb{R}$):

$$V(aX_1 + bX_2) = a^2 V(X_1) + b^2 V(X_2) + 2ab\,\mathrm{Cov}(X_1, X_2)$$

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The VOT in €/min is just the ratio of the time and cost sensitivities.

$VOT = \hat{\beta}_T / \hat{\beta}_C = 1/6\,€/\mathrm{min} = 10\,€/\mathrm{h}$. The Logit parameters are approximately the Probit parameters multiplied by the standard deviation $\lambda = \pi/\sqrt{6}$ of the Gumbel distributed Logit RUs. The VOT is essentially unchanged.

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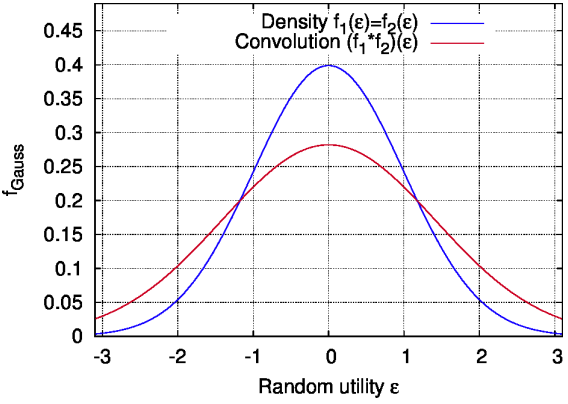
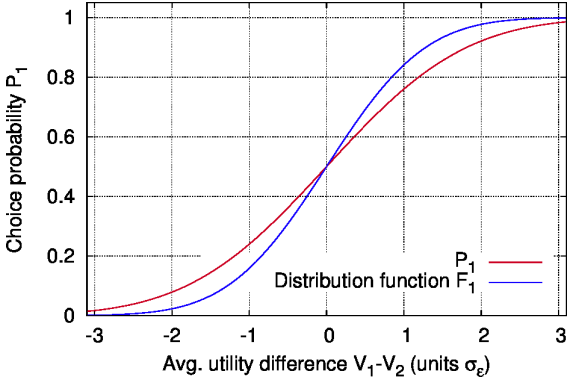
Choice probabilities of the binary Probit model II

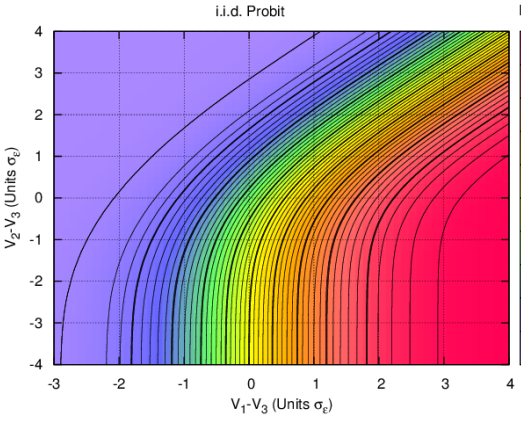


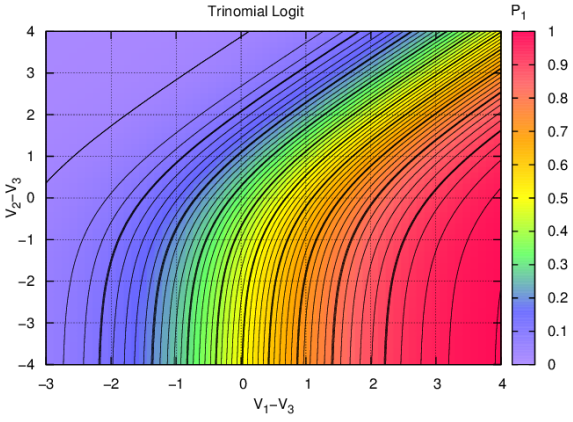
Densities of the standardnormal distributed random utilities ϵ_1 and ϵ_2 and of the utility difference $\epsilon_1 - \epsilon_2$



Distribution functions of the random utilities and the utility difference as a function of the deterministic utility difference $V_1 - V_2$

Choice probabilities of trinomial i.i.d. Probit and Logit





Symmetrie considerations (for P_2, swap $1 \leftrightarrow 2$):

$$\begin{aligned} P_2(V_1 - V_3, V_2 - V_3) &= P_1(V_2 - V_3, V_1 - V_3), \\ P_3 &= 1 - P_1 - P_2 \end{aligned}$$

8.3 Elasticities

- *General definition:*

Elasticities denote the percentaged change of endogenous variables y_i per small percentaged change of exogenous variables x_j for an average situation, colloquially: **" % change of y per 1 % change of x"**

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  Generally, with several endogenous variables, one distinguishes between
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Since elasticities describe average aspects, we take the choice probabilities P_i rather than the discrete actual choices as endogenous variables. For the general deterministic utilities with generically modeled attributes[1]

$$V_{ni} = \sum_{m'} \beta_{m'} x_{nm'i}$$

we derive the

- **Proper (substitution) elasticities:** The attribute (characteristic) m of an alternative i feeds back on the demand for this alternative:

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? Derive the formulas for the proper and cross elasticities

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$$j = i\text{: } \epsilon_{nii} = \beta_m x_{nmi} \left(1 - P_{ni}\right), \quad j \neq i\text{: } \epsilon_{nij} = -\beta_m x_{nmj} P_{nj}$$

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(Notice $\sum_i P_{ni} = 1$ in the last step!)

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Motivation: a change of characteristic m in alternative j shifts the preferences. However, they must be shifted somewhere (completeness! no option outside of the alternative set available!), so the net change is zero (notice that $P_{ni}\epsilon_{nij}^{(\mathsf{m})} = x_{nmj}\frac{\partial P_{ni}}{\partial x_{nmj}}$ and x_{nmj} is constant for the summation over i)

Questions (3)

? The cross elasticities do not depend on i, i.e., on the target alternative for the changing demand. Motivate this by the IIA condition

According to the IIA, if the utility of a "third" alternative j changes, the relative preferences $P_{ni}/p_{ni'}$ of all other alternatives $i, i' \neq j$ remains unchanged, i.e., a change of alternative j *simultaneously* changes all other alternatives such that the ratios remain the same. This is only possible if the relative changes of all probabilities $P_{ni}, i \neq j$, i.e., the cross elasticities, are the same.

? Given are three airports i from which person n can book flights to a desired destination at cost C_{ni} (because of revenue management, C depends on n), so

$$V_{ni} = \beta_{01}\delta_{01} + \beta_{02}\delta_{02} + \beta_1 C_{ni}$$

Show that the proper elasticities are negative while the cross elasticities are positive.

Proper elasticity $\epsilon^{(C)}_{nii} = \beta_1 C_{ni}(1 - P_{ni}) < 0$ since $P_{ni} < 1$ $C_{ni} > 0$, and the price sensitivity $\beta_1 < 0$. The cross elasticities $\epsilon^{(C)}_{nii} = -\beta_1 C_{nj} P_{nj}$ are therefore positive.

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8.3.2 Macroscopic elasticities

For a company, the relative probability increase of single customers chosing their products is not relevant but the aggregate over all customers. Hence, the macroscopic elasticity

$$\epsilon_{ij}^{(\text{mac,m})} = \frac{X_{mj}}{N_i}\frac{\partial N_i}{\partial X_{mj}}, \quad X_{mj} = \sum_{n=1}^{N} x_{nmj}, \quad N_i = \sum_{n=1}^{N} P_{ni}$$

gives the percentage increase of people chosing alternative i when the sum of attribute m over all persons booking a flight increases at alternative j by one percent.

(i) Same absolute changes for all persons, $\mathrm{d}x_{nmj} = \mathrm{d}X_{mj}/N$:

$$\epsilon_{ij}^{(\text{mac,abs,m})} = \frac{X_{mj}}{N_i}\,\frac{1}{N}\sum_{n}\frac{P_{ni}}{x_{nmj}}\epsilon_{nij}^{(\text{mic,m})}$$

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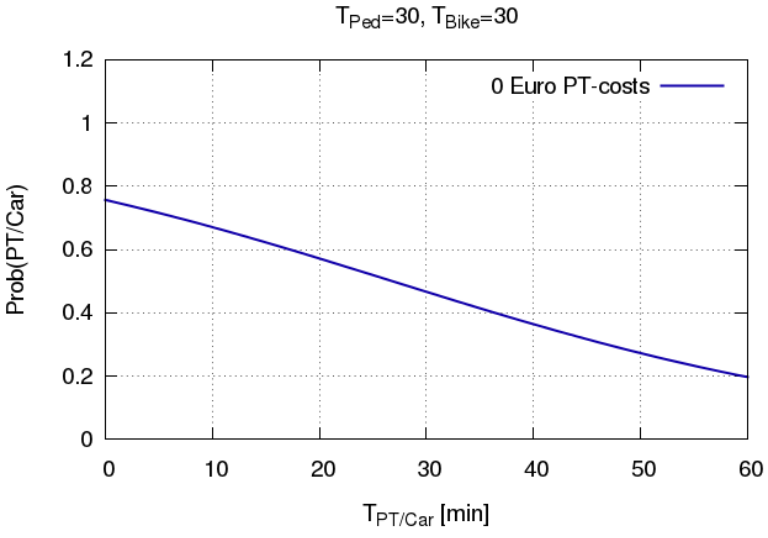
Appendix

Model1: Dependence of the modal split on the PT attributes



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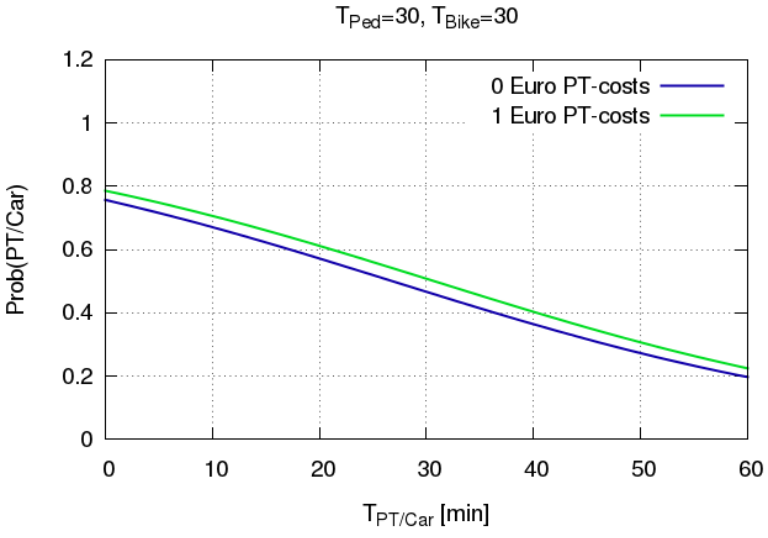

Wrong sign for cost sensitivity, too low time sensitivity!

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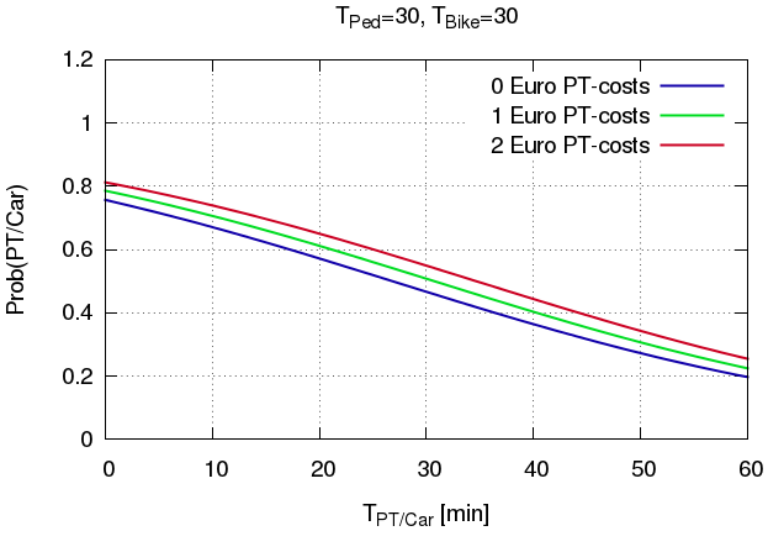

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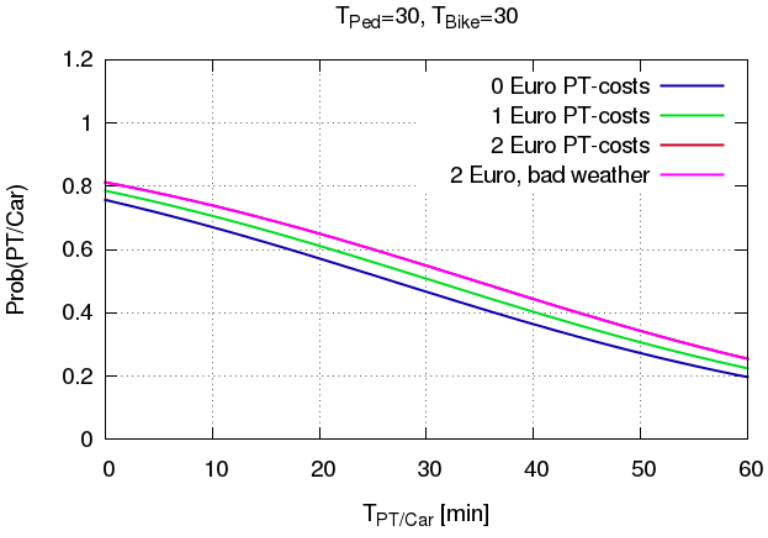

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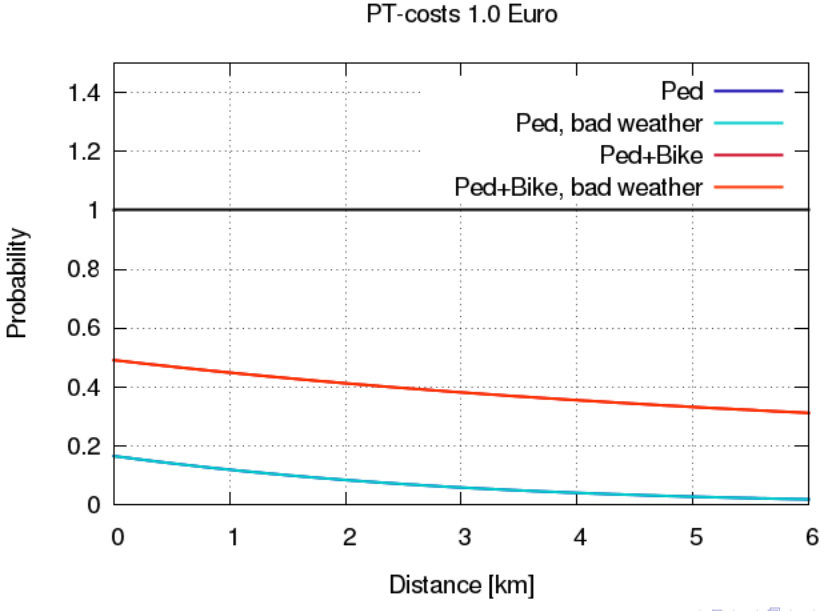
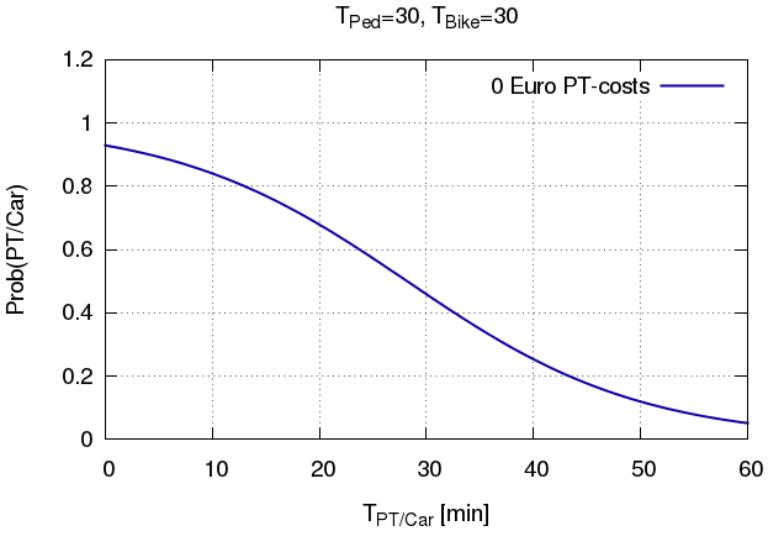
Model 1: Dependence on the distance

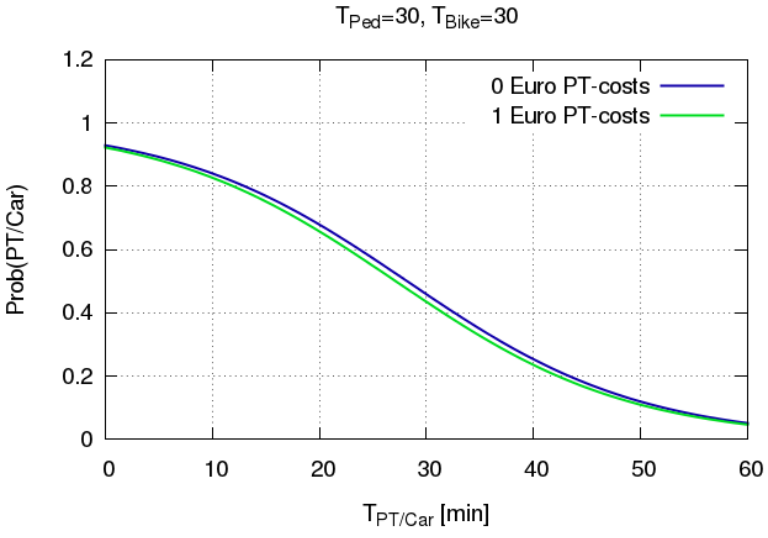
assuming plausible speeds 5, 15, and 25 km/h for each mode, respectively



Model 2: Dependence of the modal split on the PT attributes

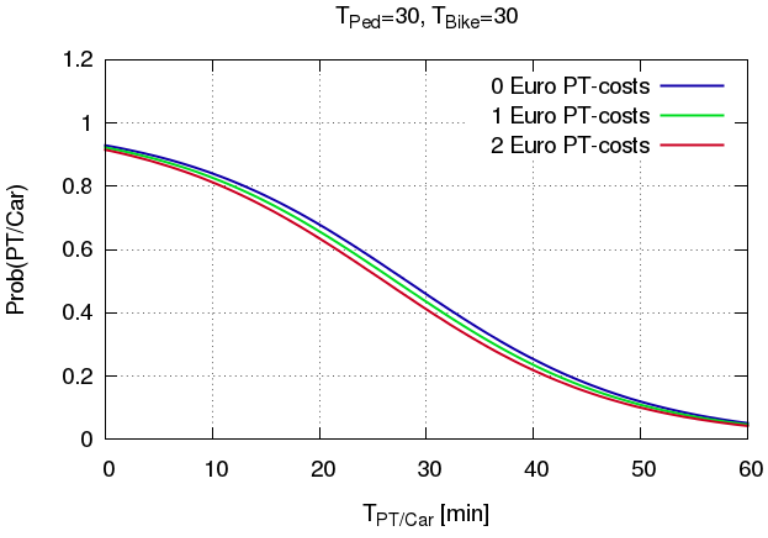


Model 2: Dependence of the modal split on the PT attributes



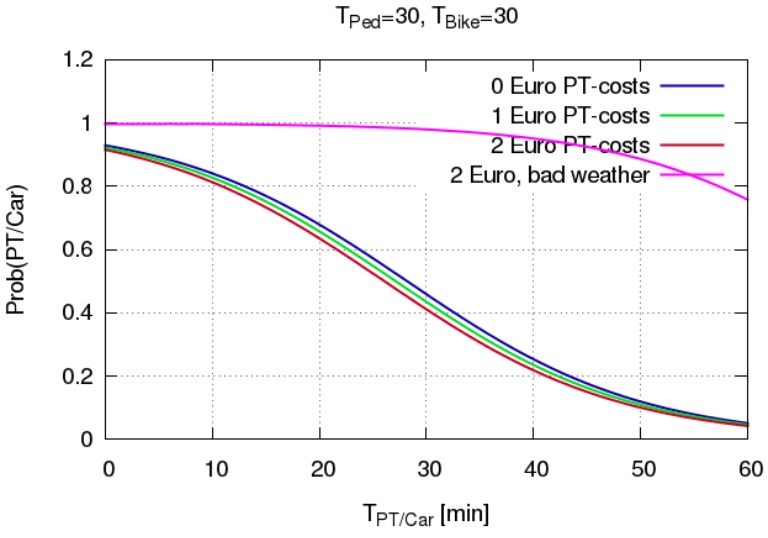
Too low cost sensitivity!

Model 2: Dependence of the modal split on the PT attributes

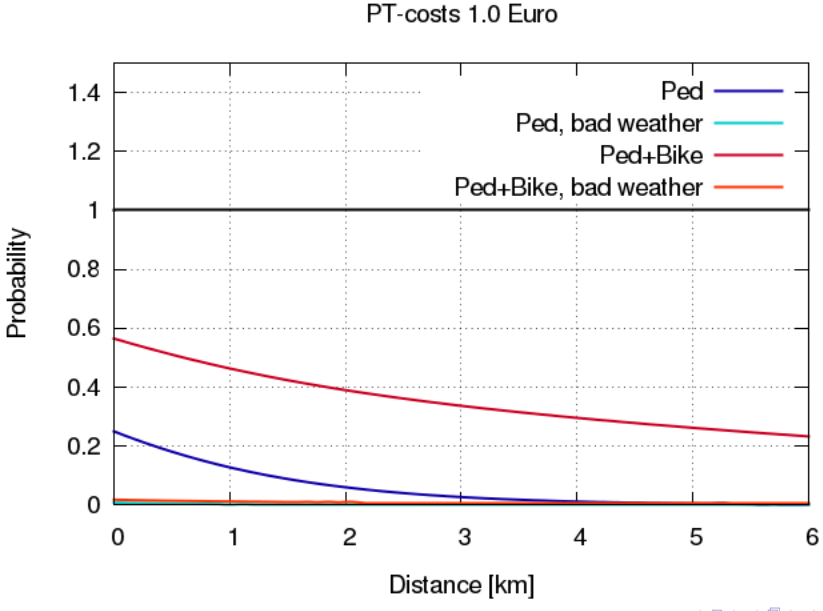


Too low cost sensitivity!

Model 2: Dependence of the modal split on the PT attributes



Too low cost sensitivity!

Model 2: Dependence on the distance

assuming plausible speeds 5, 15, and 25 km/h for each mode, respectively