Lecture 02: Linear (Regression) Models

2.1 Flow Chart of the Econometric Method

2.2 Model Specification

2.2.1 Functional specification

2.2.2 Statistical specification

2.2.3 Data specification

2.3 Ordinary Least Squares (OLS) Estimation

2.1 Flow Chart of the Econometric Method

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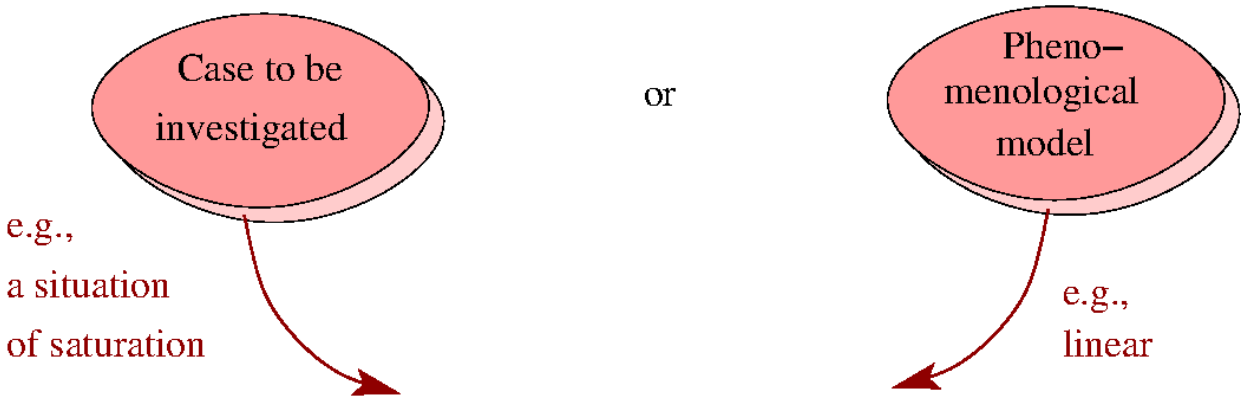

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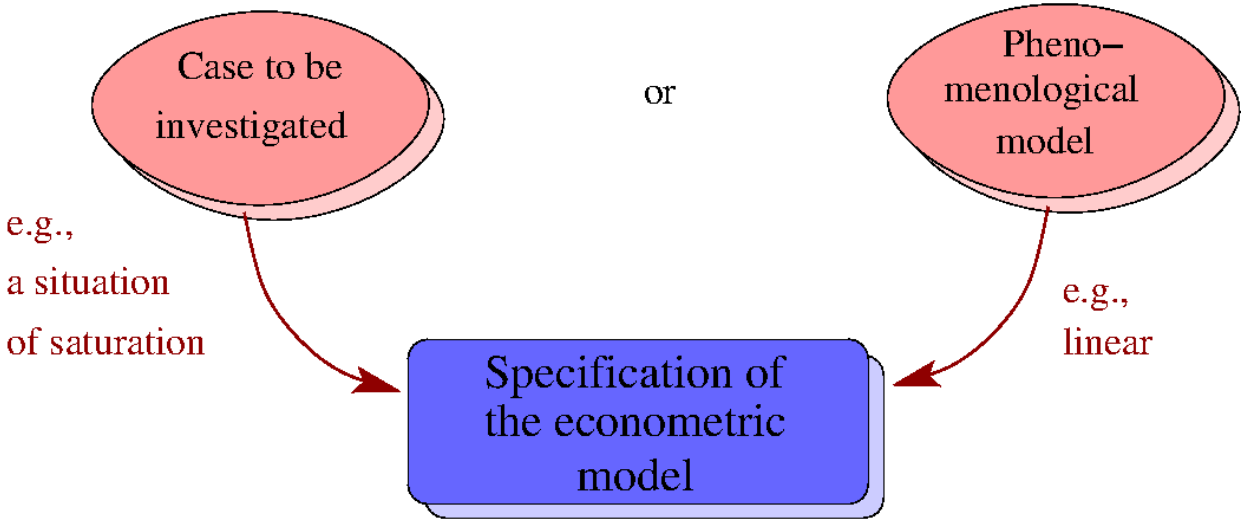

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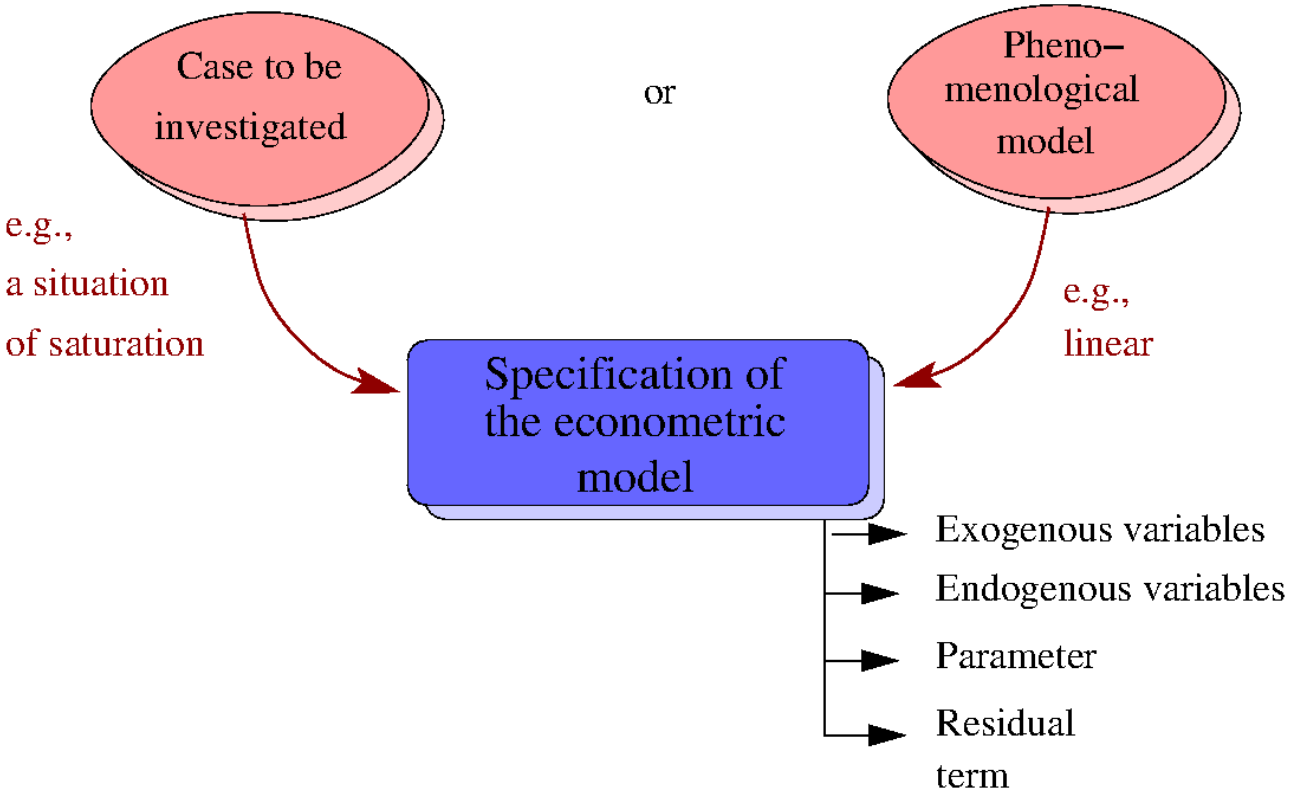

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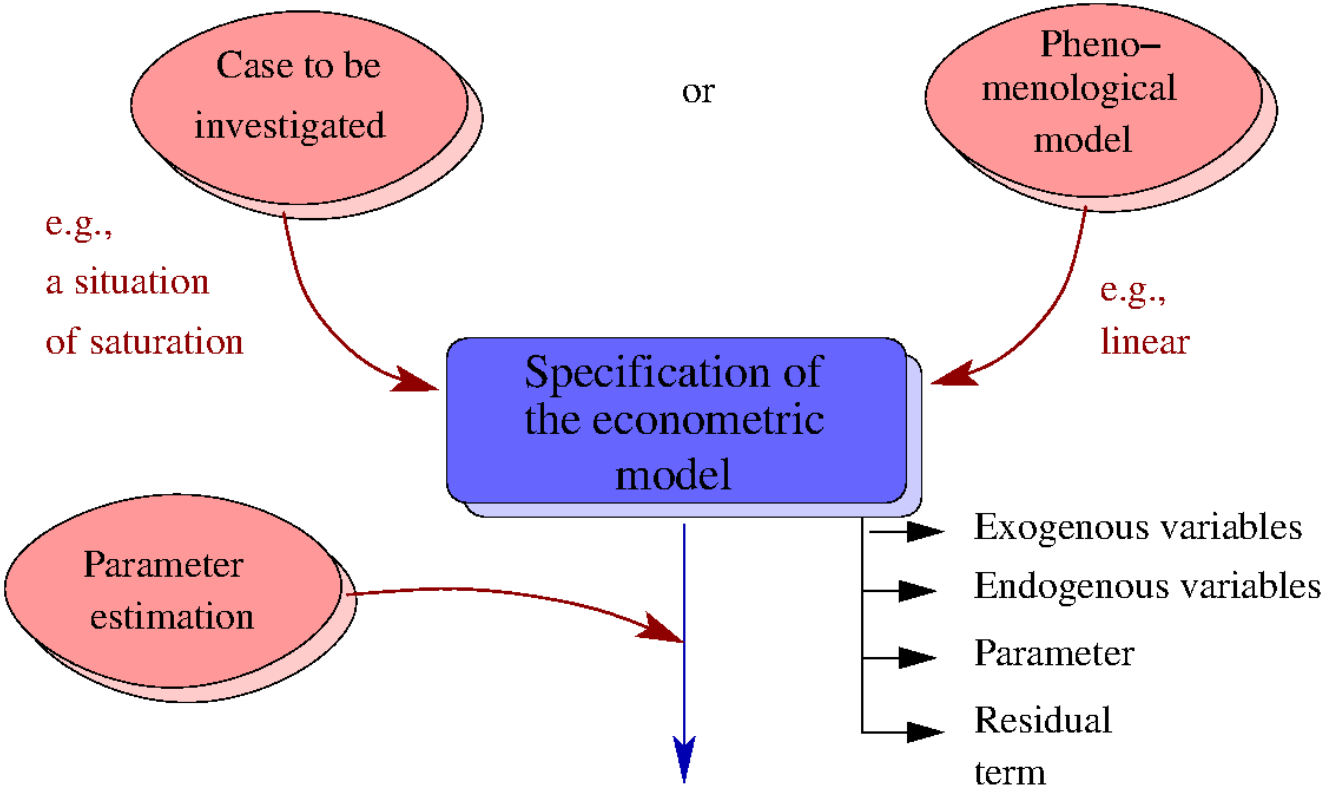

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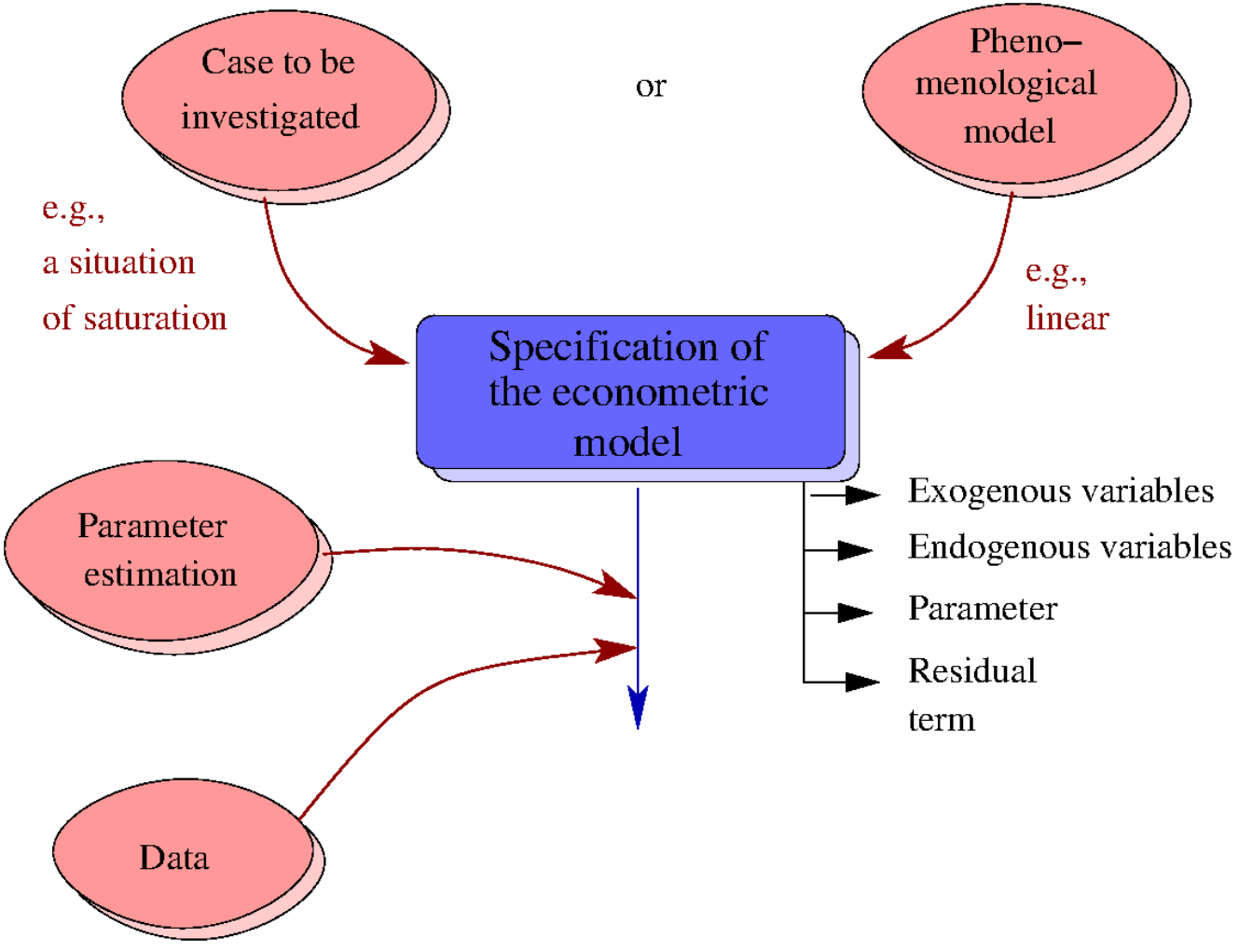

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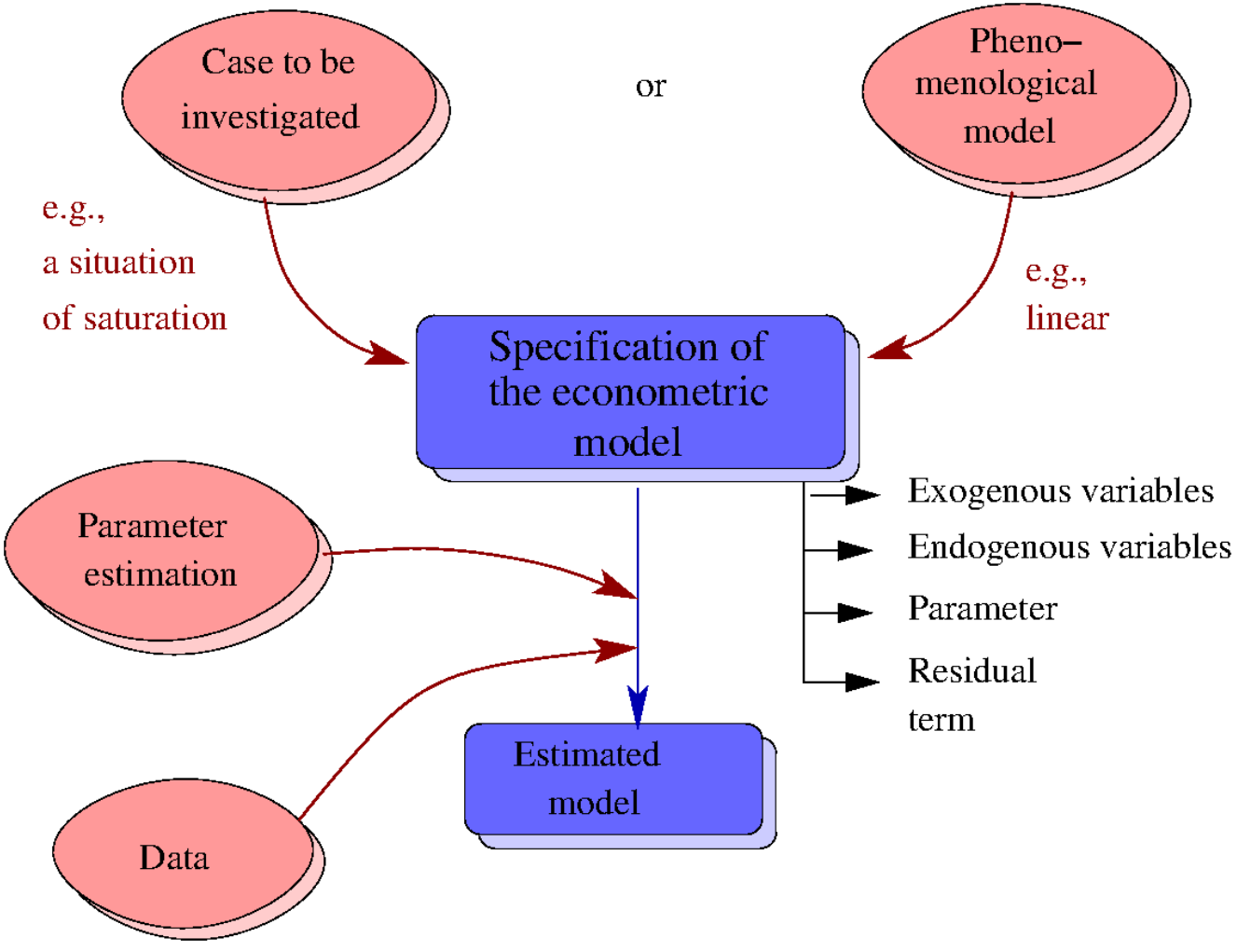

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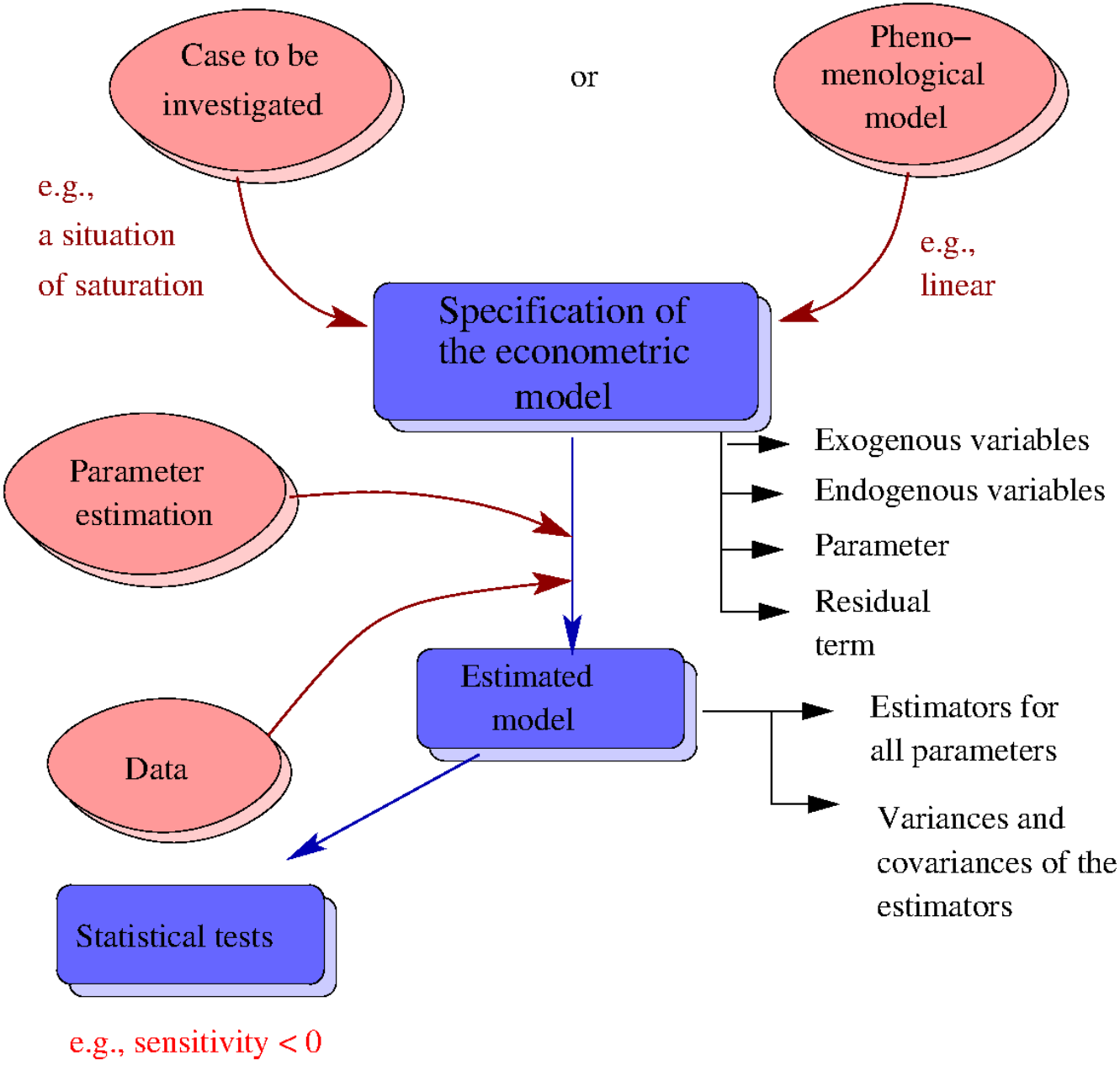

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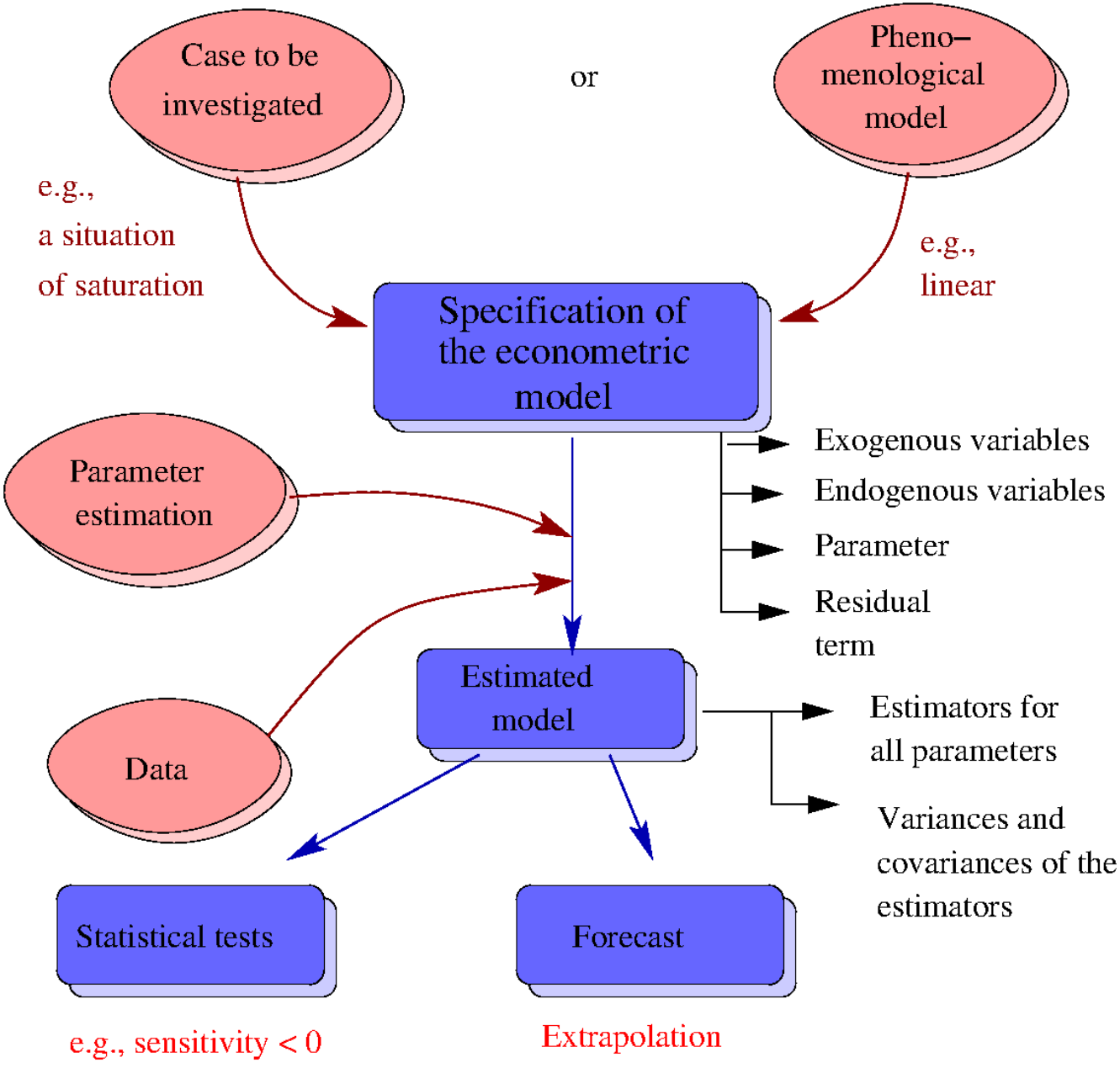

2.2. General Model Specification

As **model specification**, we denote the *complete structural definition* of the model and its consistency with the available data. There are three aspects:

- **Functional specification**: The model's exogenous and endogenous variables and the functional form in which they appear. In **linear models**, this means the specification of the fixed, generally nonlinear functions $x_j = g_j(\tilde{\boldsymbol{x}}))$ mapping the original exogenous variables $\tilde{\boldsymbol{x}}$ to the linear **factors** $x_j = g_j(\tilde{\boldsymbol{x}})$
- **Statistical specification**: If the model contains stochastic elements, e.g., residual "error" terms we want to know how they are distributed and correlated with each other
- The **data specification** should ensure that the available data can be used to analyze the data, for example, sufficient number of data sets, check if each set contains all the exogenous and endogenous variables

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WARNING

If the econometric model is not specified correctly, all sorts of problems occur, from irrelevant to nasty:

- irrelevant: some mis-specification are detected automatically during model estimation producing "zero/zero" errors and the like, or even self-corrected.
- mild: a mis-specification is not detected automatically but there is no bias and the estimation method is even efficient. However, inferential conclusions may be incorrect
- medium: the results are still unbiased but the inferential analysis is not efficient and generally gives erroneous conclusions (higher significance than in reality)
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Junk in, junk out!

There are lies, damned lies, and **statistics!**

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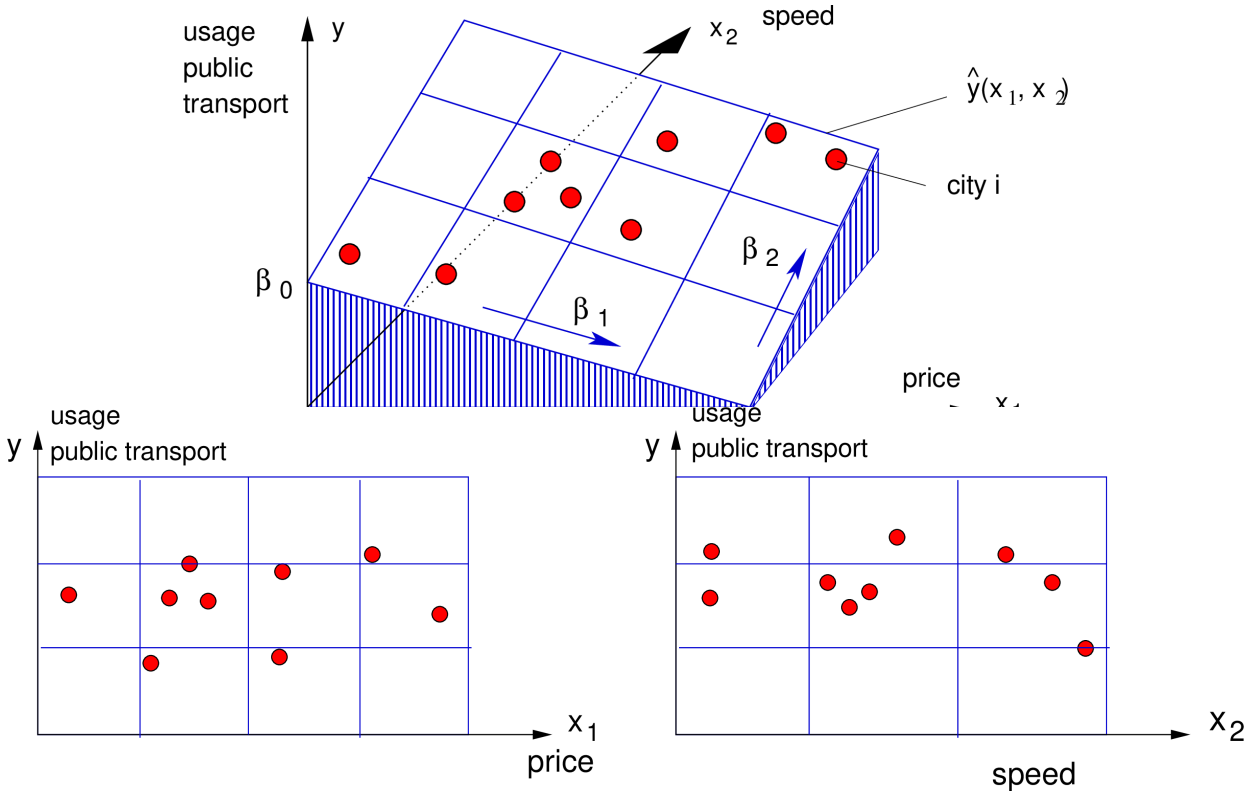
Linear models

The general (parameter-) linear model has the form

$$y = \sum_j \beta_j x_j + \epsilon = \sum_j \beta_j g_j(\tilde{\boldsymbol{x}}) + \epsilon$$

- x_j, $j = 0, ..., J$ are the $p = J + 1$ linear **factors**
- The functional specification defines the fixed nonlinear functions $g_j(\tilde{\boldsymbol{x}})$ mapping the vector $\tilde{\boldsymbol{x}}$ of exogeneous variables to the vector $\boldsymbol{x}$ of factors
- Often, but not always, the first factor $x_0 = 1$ such that the associated parameter denotes the **intercept**. Give an example where no intercept is allowed Number of cars of the city residents as a function of the city size
- The number of factors is generally not equal to the number of exogenous variables. Explain this for the simple model $y = a + b\tilde{x} + \epsilon$
- Each of the $p = J + 1$ factors is associated with a parameter, so the model is both linear with respect to the p factors and with respect to its p parameters

2.2.1 Functional specification 1: relevant factors



- All relevant influencing factors should be taken into account (top), no one missed (bottom).
- **Consequences of missing factors:** a **bias**, i.e., **"junk in, junk out"**
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- **Solution:** check for superfluous factors: *F-test*; finding missing factors: *your expertise!*

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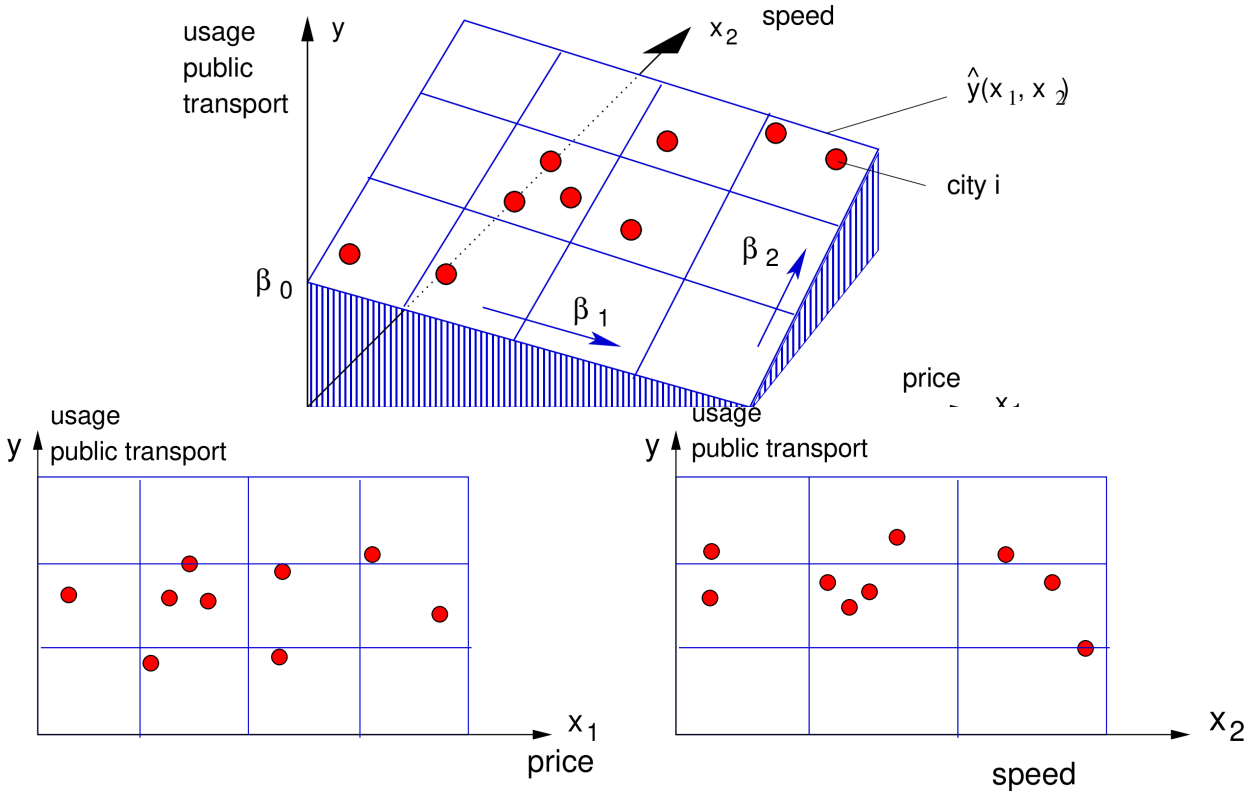

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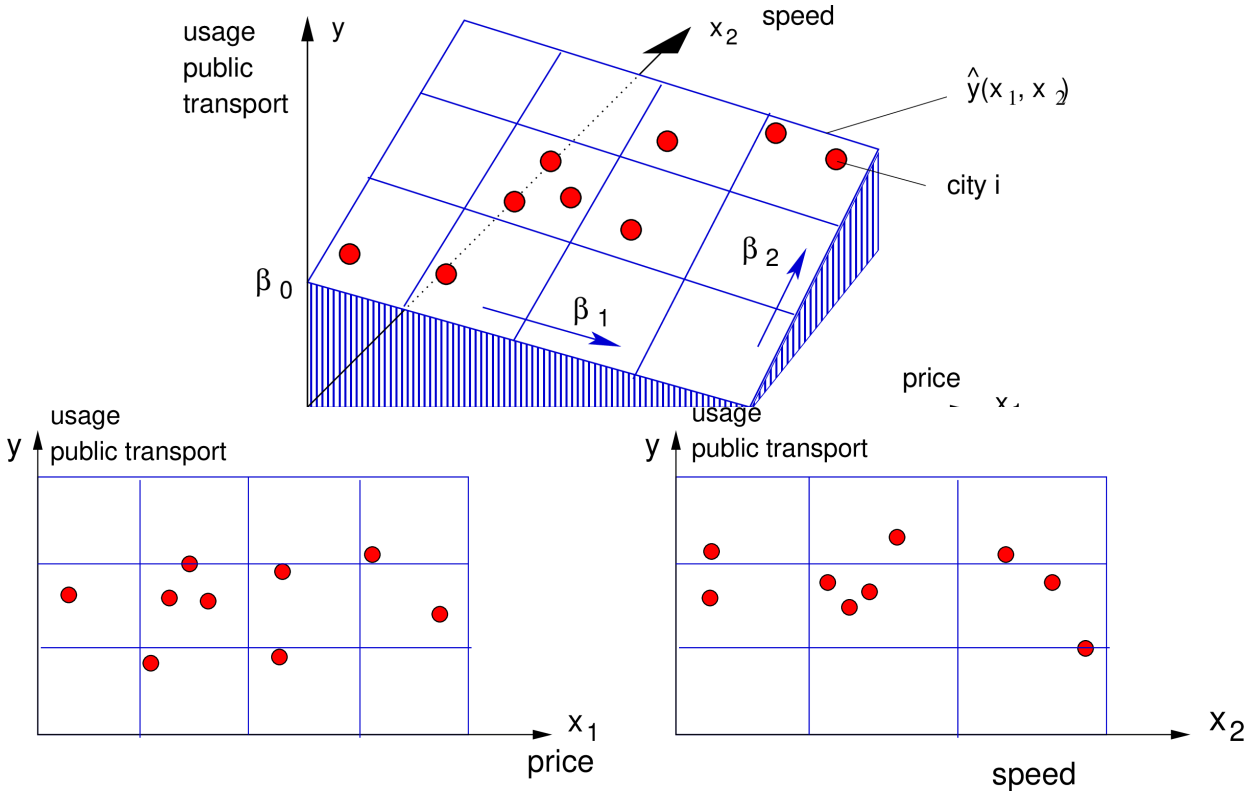

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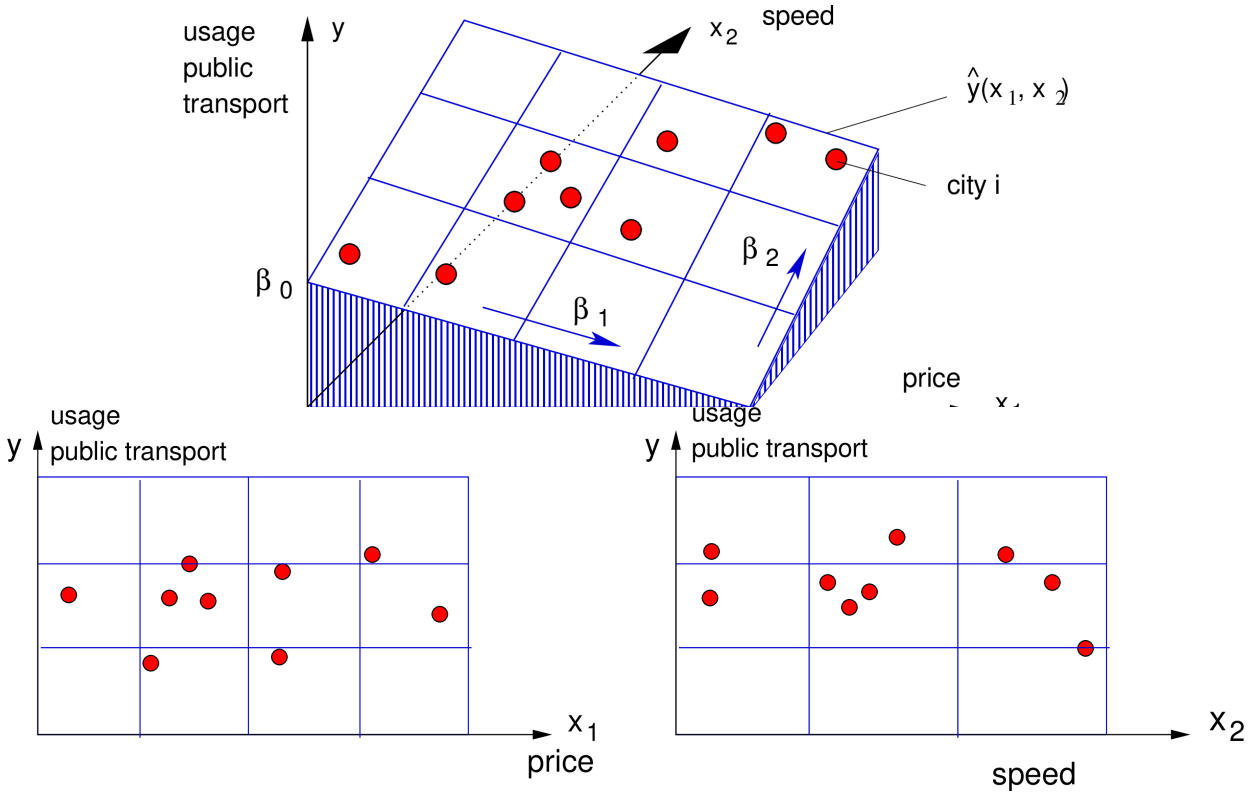

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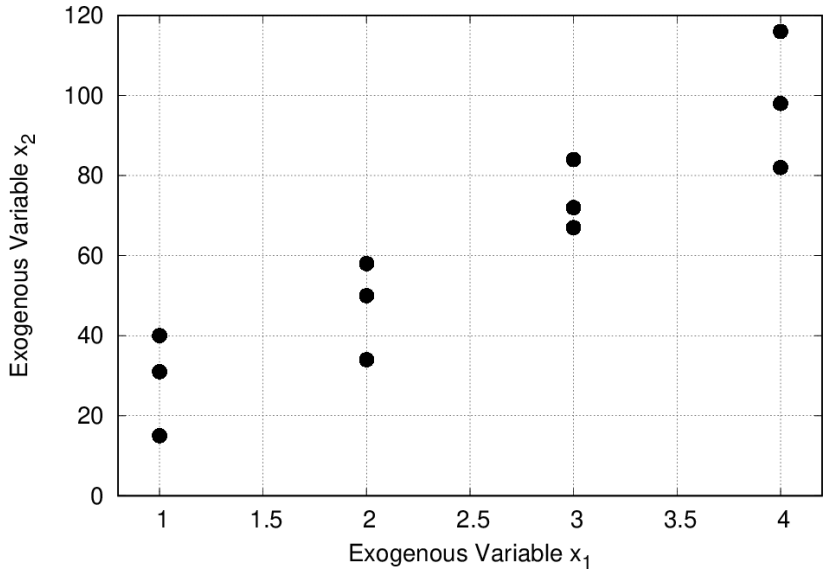
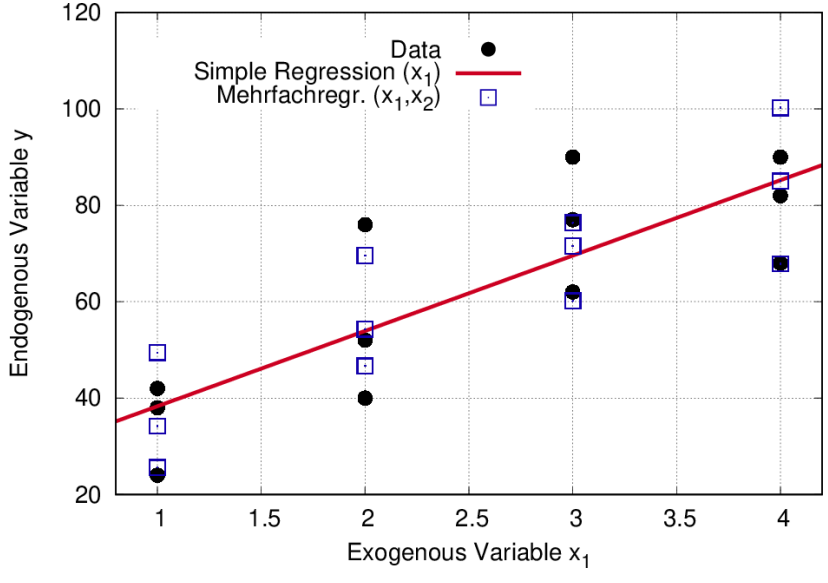
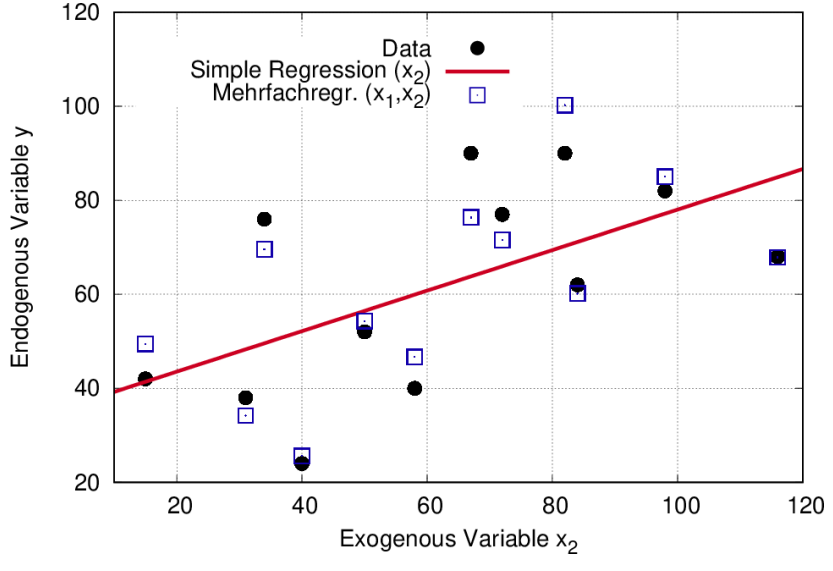
Example: modeling the demand for hotel rooms

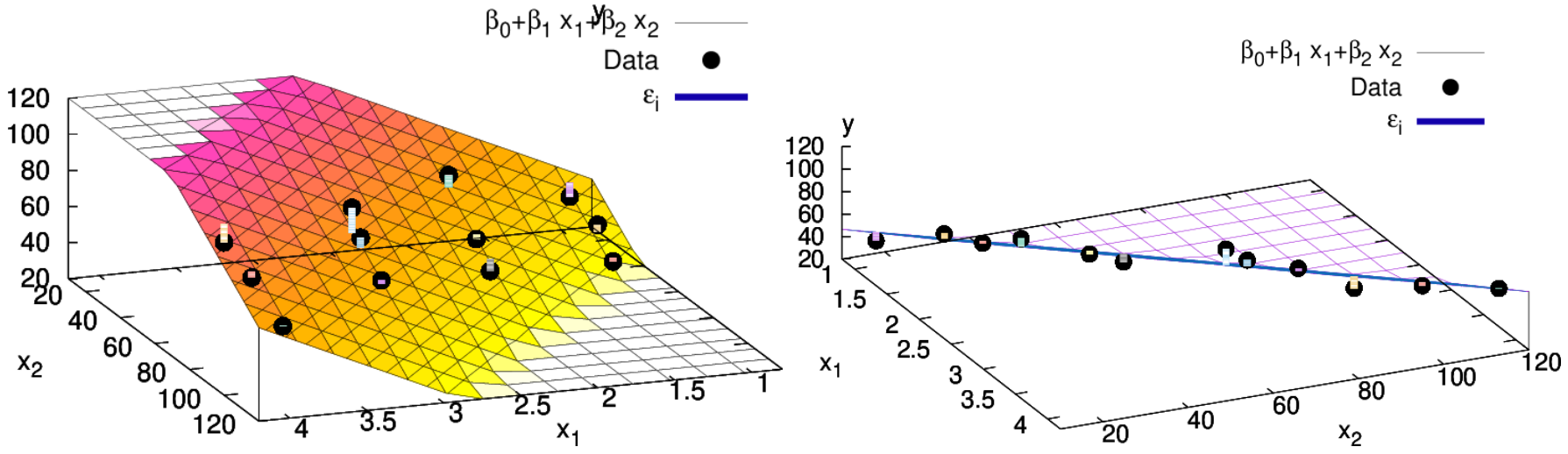


- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$ with the factors: $x_0 = 1$, x_1: proxy for quality [# stars]; x_2: price [€/night].
- The exogenous variables/factors are non-perfectly correlated: ✓
- Endogenous variable: booking rate [%]
- The demand is positively correlated with both the quality and the price (!)





Visualization of the fit quality



- Surface: model $\hat{y}(\boldsymbol{x}) = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$
- Black bullets: data (right graphics: twice mirrored)
- OLS estimate: $\hat{\beta}_0 = 25.5$, $\hat{\beta}_1 = 38.2$, $\hat{\beta}_2 = -0.953$.
- Blue or pink bars: residuals ϵ_i (≤ 0 if below the model plane)

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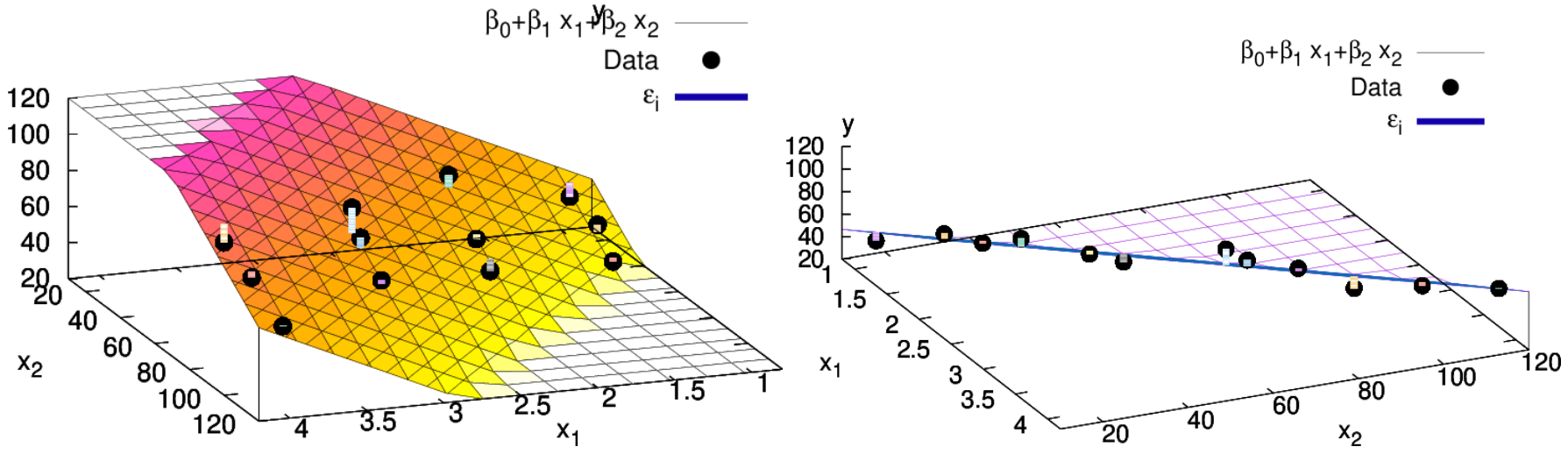

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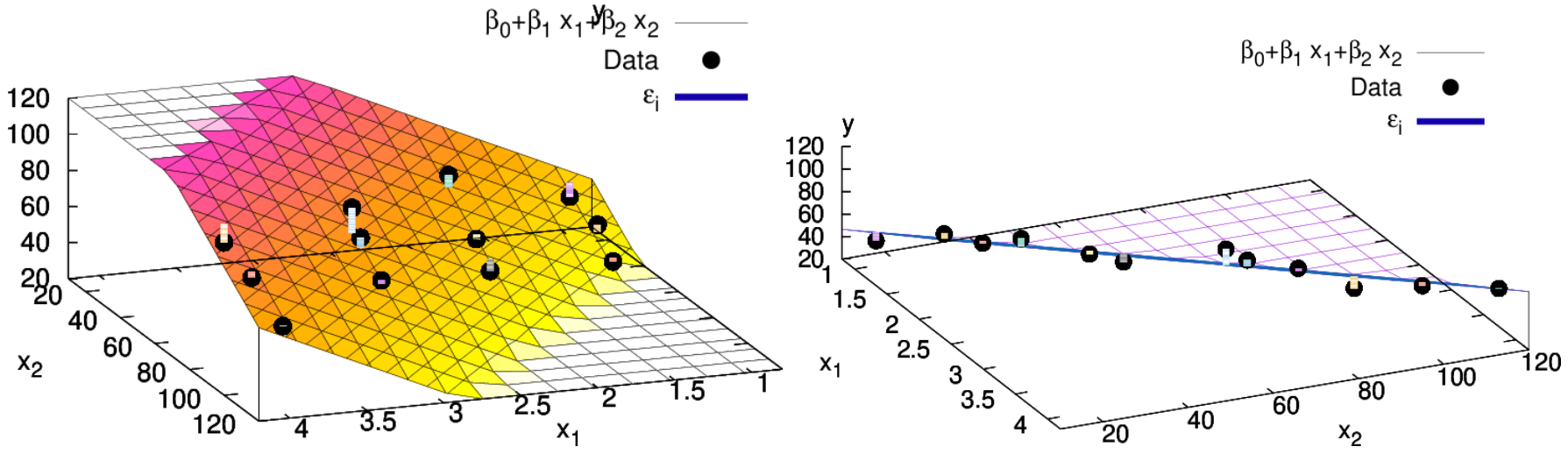

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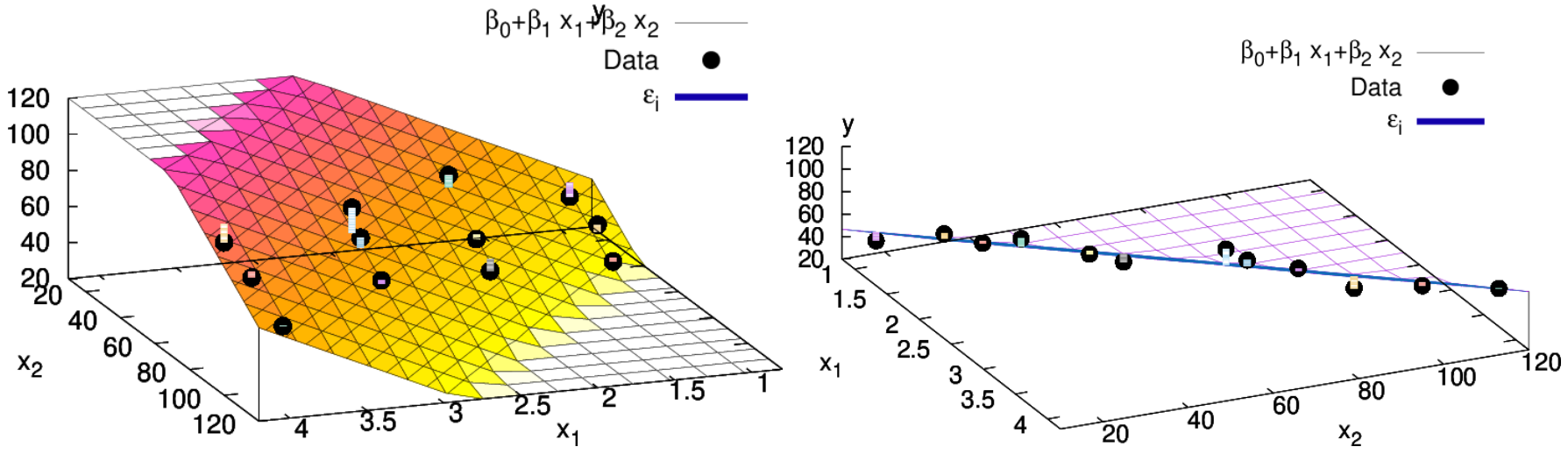

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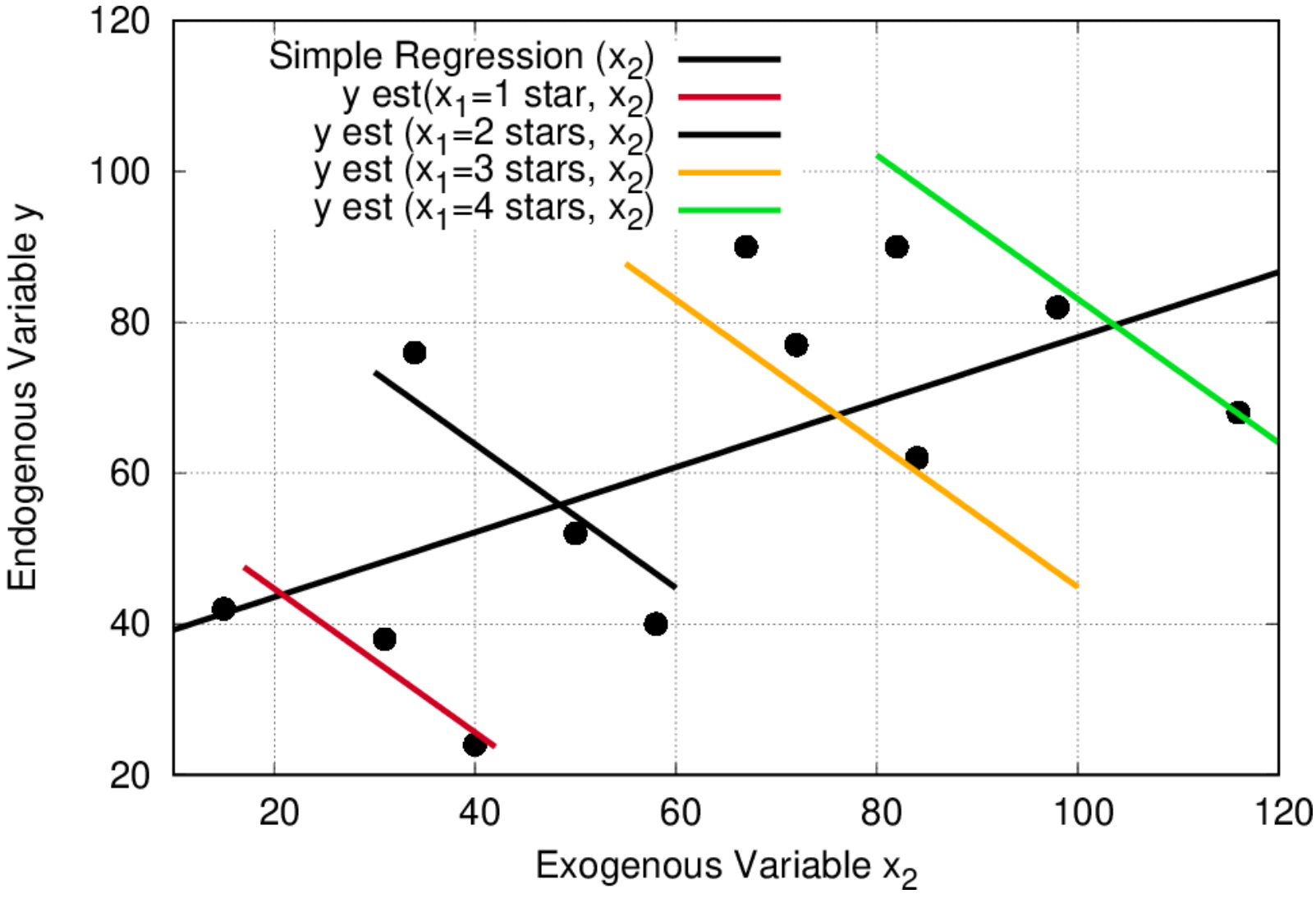
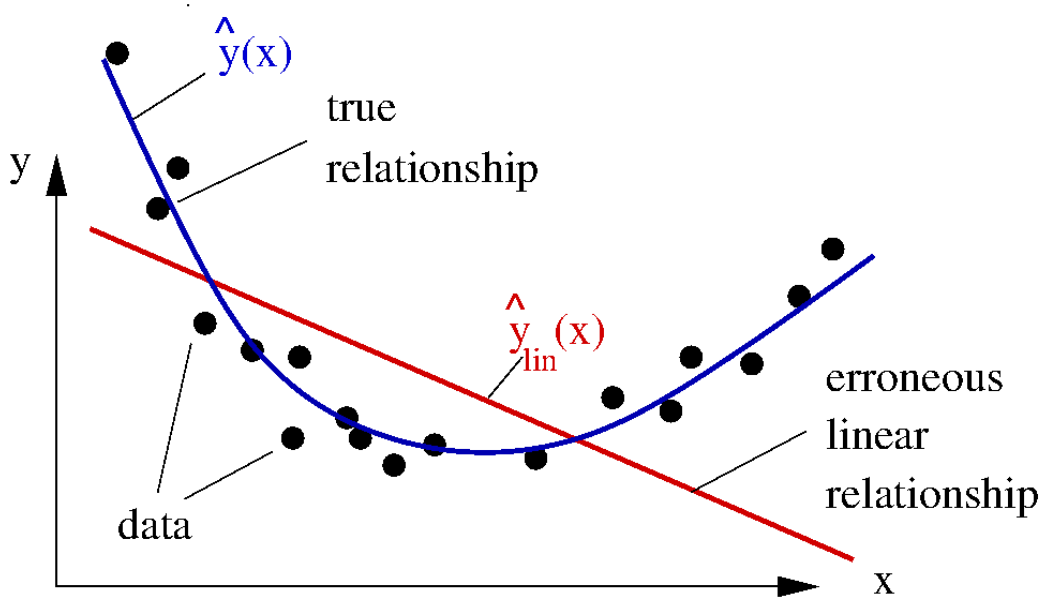
Effect of the correlations between the exogenous variables



Functional specification 2: linearity



- The model should be linear which is not fulfilled here.
- **Consequences of violation:** "junk in, junk out"
- **Solution:** A change of the independent variable into several **factors** would be a solution here, e.g. $x'_0 = 1, x'_1 = 1/x, x'_2 = x^2$ or $x'_0 = 1, x'_1 = x, x'_2 = x^2$.

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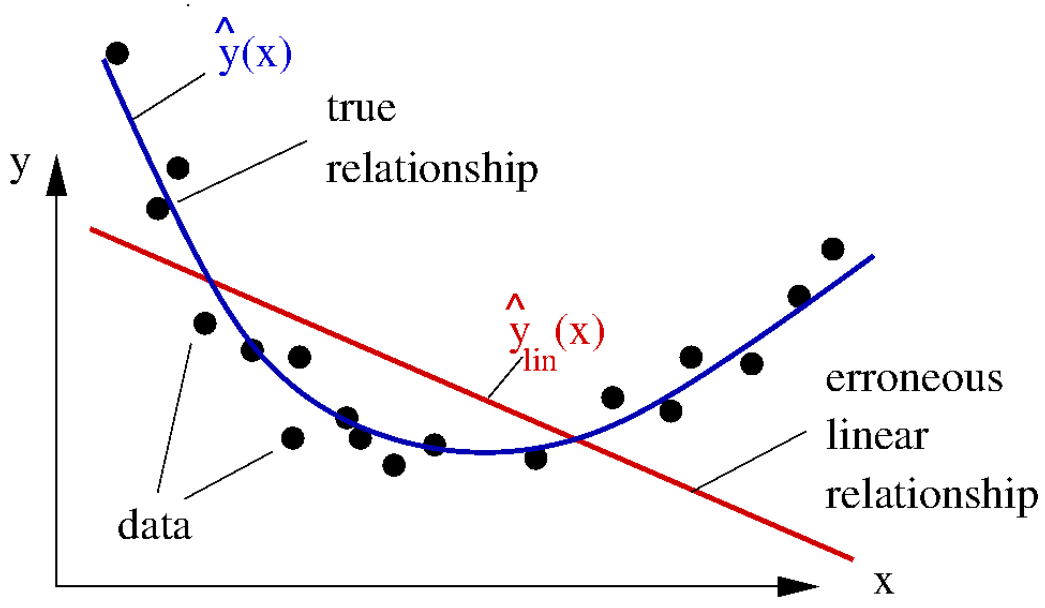

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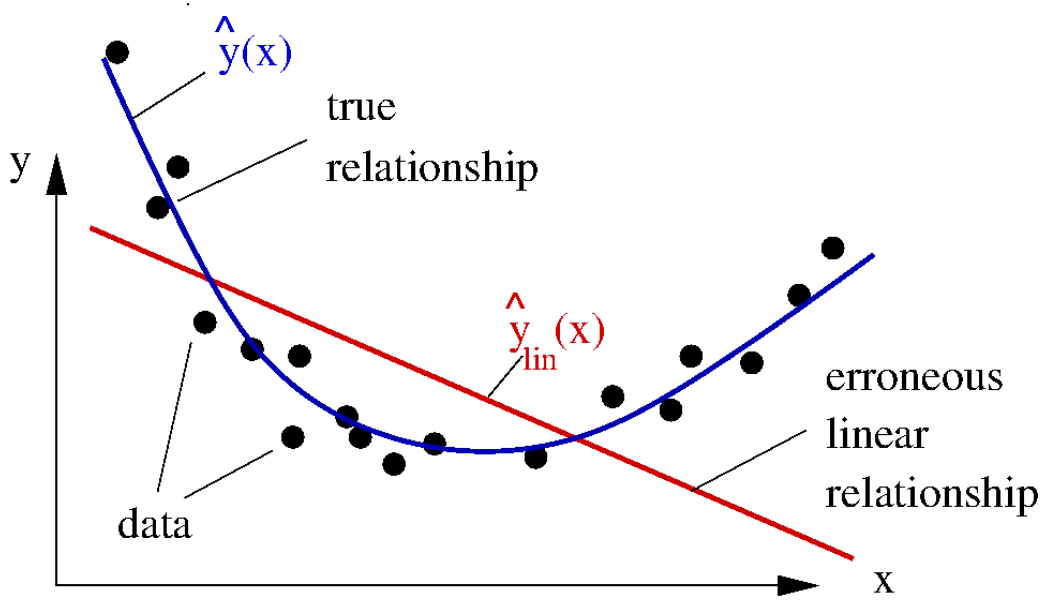

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Example: fuel consumption

Assuming a constant efficiency chemical energy $\rightarrow$ mechanical energy, the required fuel per 100 km, y, is proportional to the driving resistance with the contributions

- Friction tire-road: contributions independent of the speed $\tilde{x}_1$ and proportional to the mass $\tilde{x}_2$.
- Air drag: proportional to speed squared, $\tilde{x}_1^2$, and independent from mass
- Gradient: proportional to mass times gradient $\tilde{x}_3$

In addition, there is a base consumption rate (about 0.6 liters/h) when the car is idling/driving very slowly $\Rightarrow$ contribution proportional to 1/speed [liters/km=liters/h * h/km] $\Rightarrow$ model

$$y(x) = \sum_{j=1}^{4} \beta_j x_j + \epsilon, \quad x_1 = \tilde{x}_2, \quad x_2 = \tilde{x}_1^2, \quad x_3 = \tilde{x}_2 \tilde{x}_3, \quad x_4 = \frac{1}{\tilde{x}_1}$$

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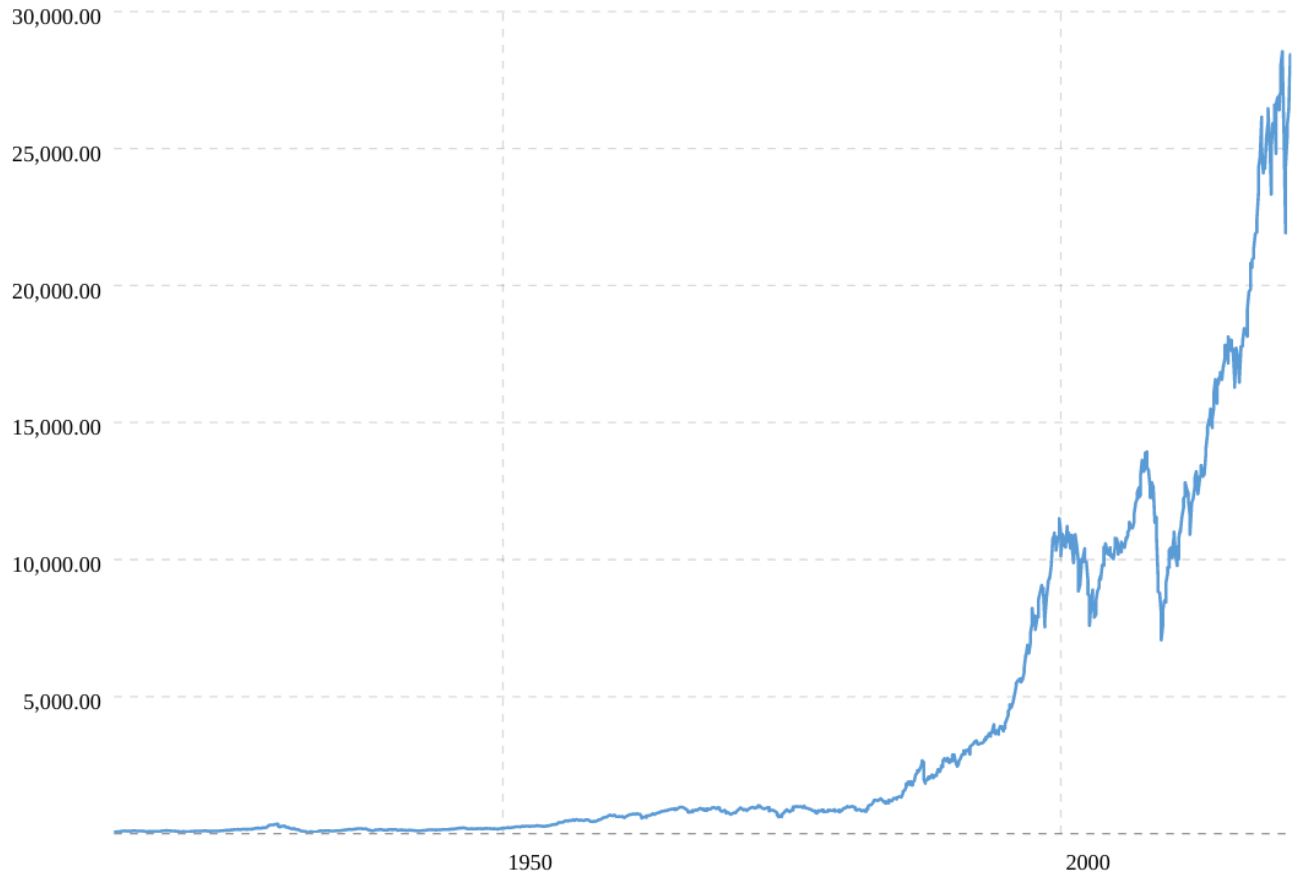
- Gradient: proportional to mass times gradient $\tilde{x}_3$

In addition, there is a base consumption rate (about 0.6 liters/h) when the car is idling/driving very slowly $\Rightarrow$ contribution proportional to 1/speed [liters/km=liters/h * h/km] $\Rightarrow$ model

$$y(\boldsymbol{x}) = \sum_{j=1}^{4} \beta_j x_j + \epsilon, \quad x_1 = \tilde{x}_2, \quad x_2 = \tilde{x}_1^2, \quad x_3 = \tilde{x}_2\tilde{x}_3, \quad x_4 = \frac{1}{\tilde{x}_1}$$

Transformation of the endogenous variable I

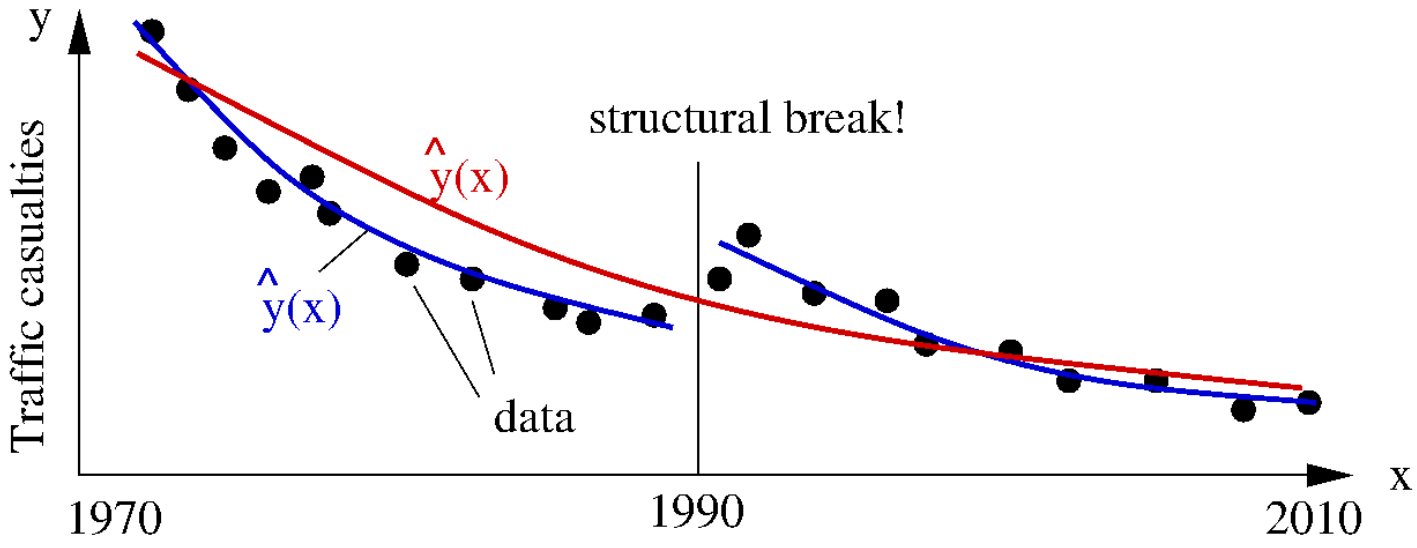


Transformation of time $\tilde{x}$ to a factor $x = \exp(\tilde{x})$ would linearize the model but the fluctuations are not i.i.d (see statistical specification below)

Transformation of the endogenous variable II

Transformation of the endogenous variable $y \to u = \ln(y)$ and $x = \tilde{x}$ gives a properly specified linear model $u(x) = \beta_0 + \beta_1 x + \epsilon$, $\epsilon \sim$ i.i.d.

Functional specification 3: homogeneity



- **Consequences:** an untreated discontinuity ("structural break") in the space of the exogenous variables leads to a **bias**, i.e., **junk in, junk out**
- **Solution:** a *dummy variable* with values 0 before, 1 after the break.

? What could possibly cause a structural break?

! 1. new data basis (GDR+West Germany → Germany); 2. Redefinition of a variable (e.g., seriously injured from visit to hospital to overnight visit)

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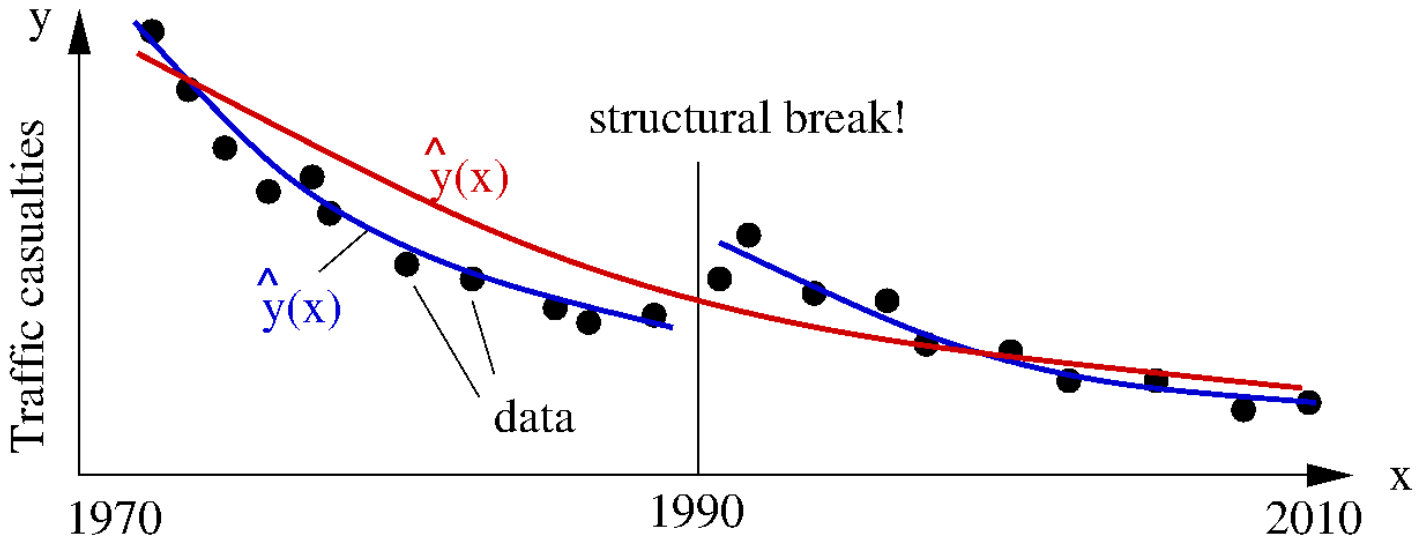

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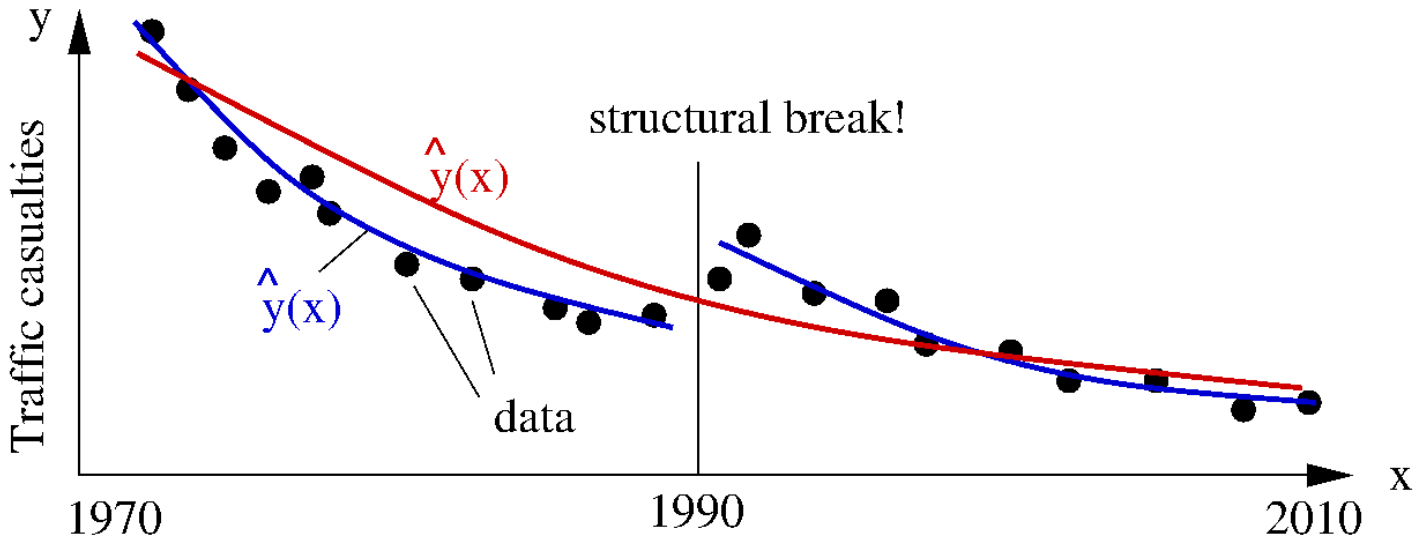

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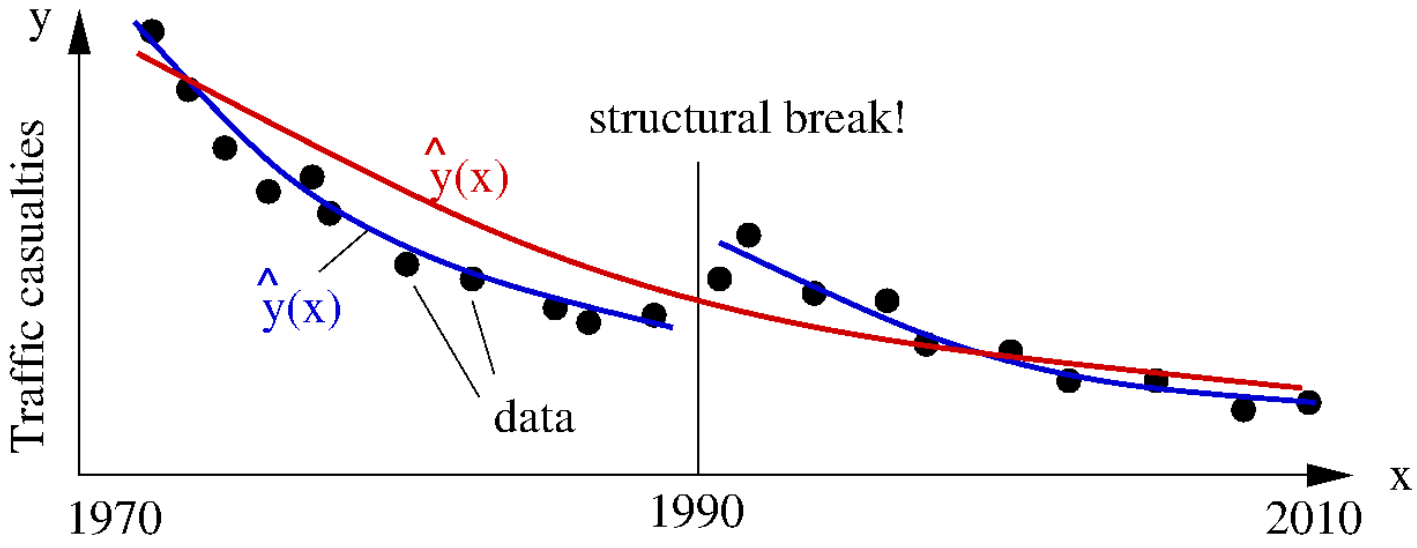

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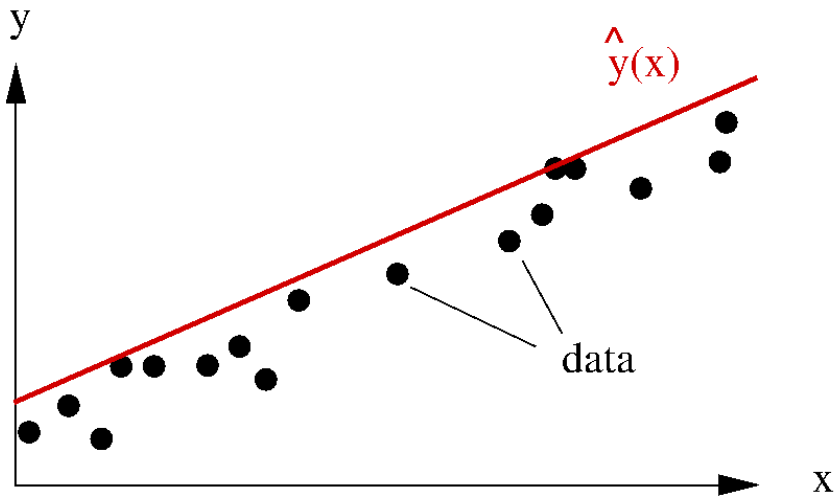
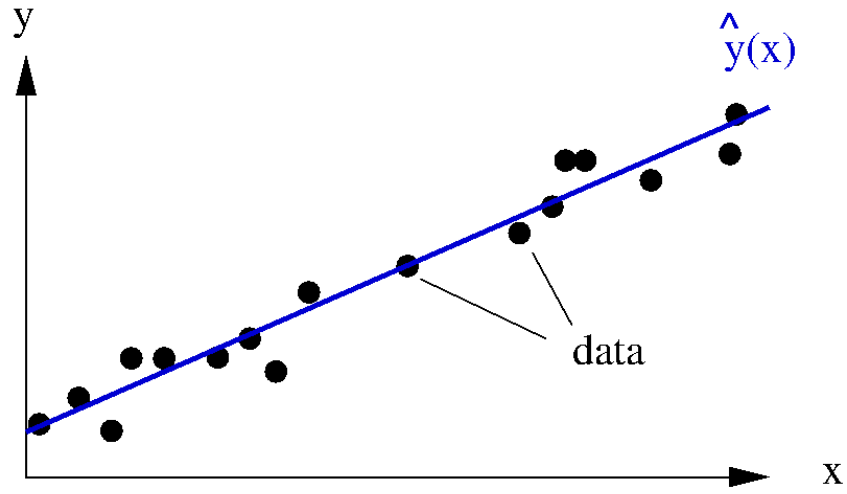
2.2.2 Statistical Specification
1. the residual ϵ has zero expectation





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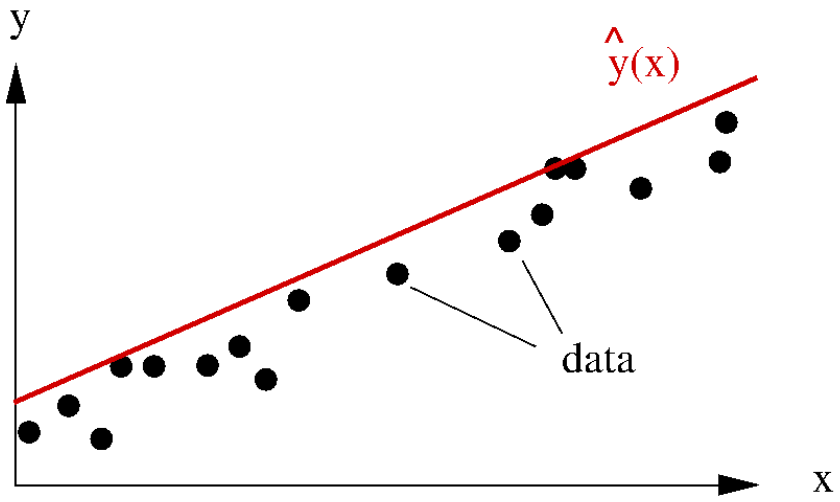

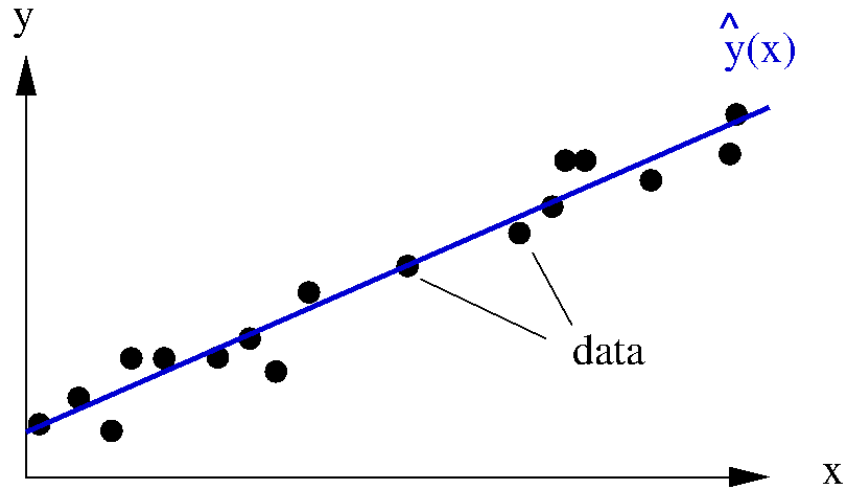

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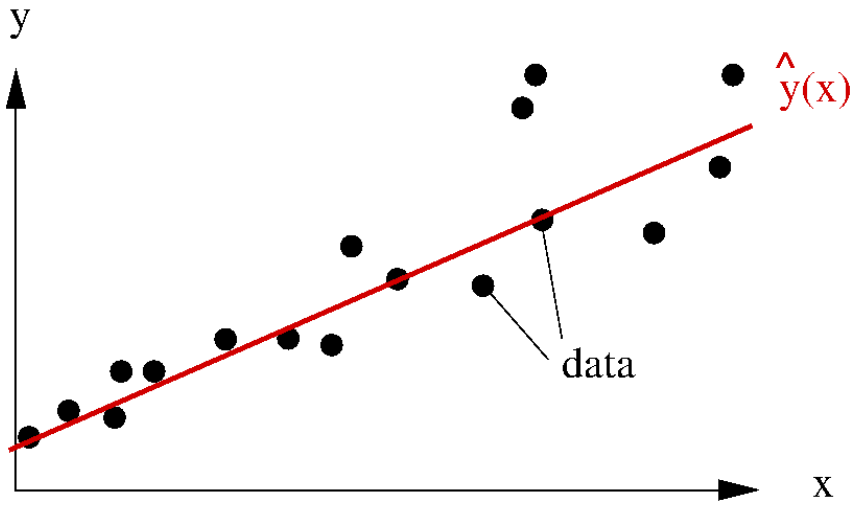

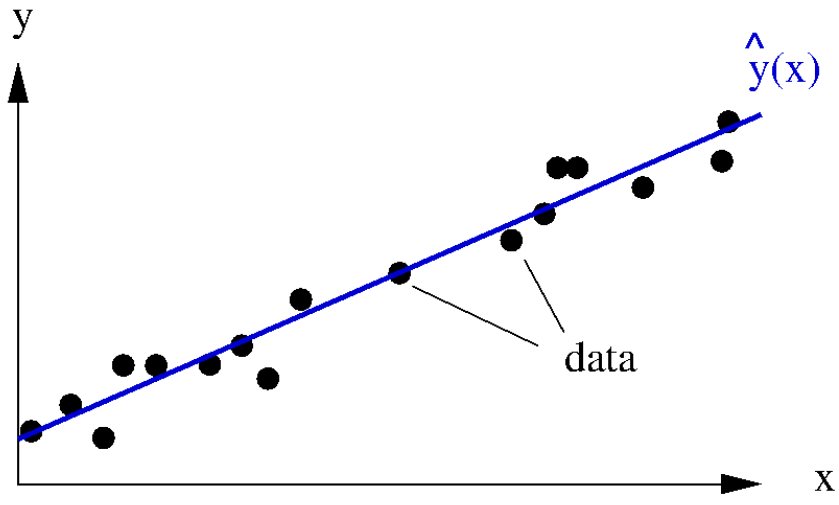

- The residual ϵ should be homoscedastic (on the right), not heteroscedastic (left).
- **Consequences:** if violated, OLS estimation remains **unbiased but is no longer efficient** (a medium error).
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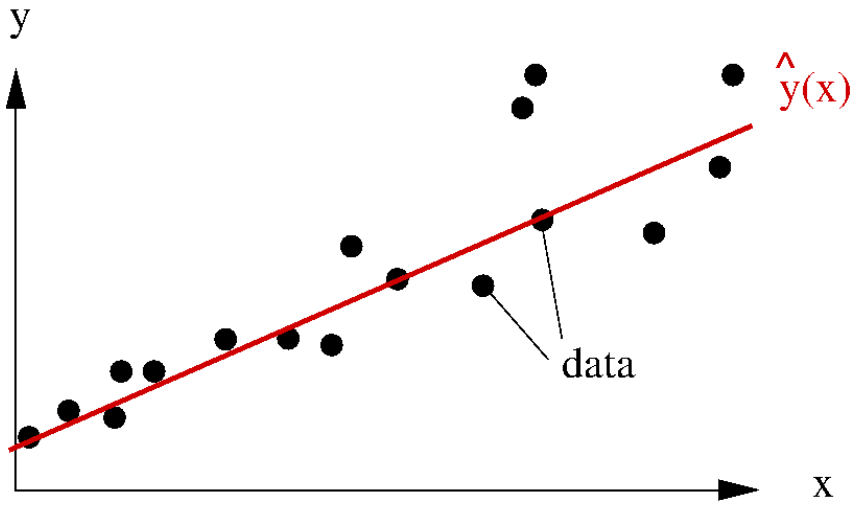

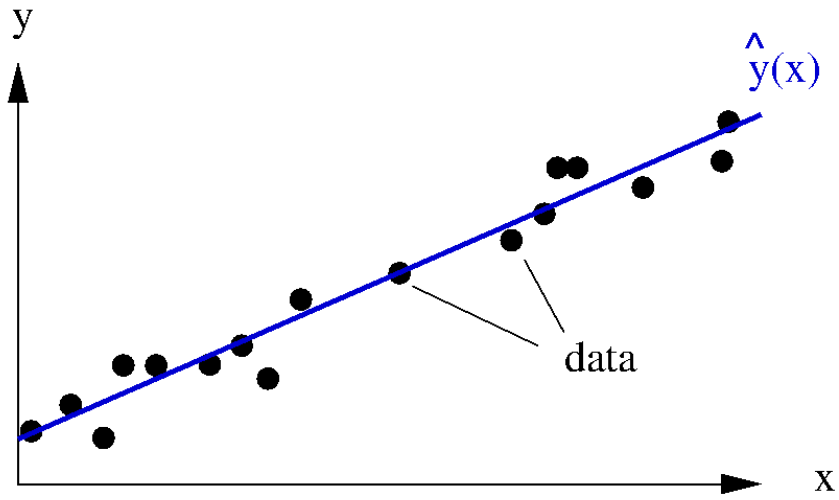

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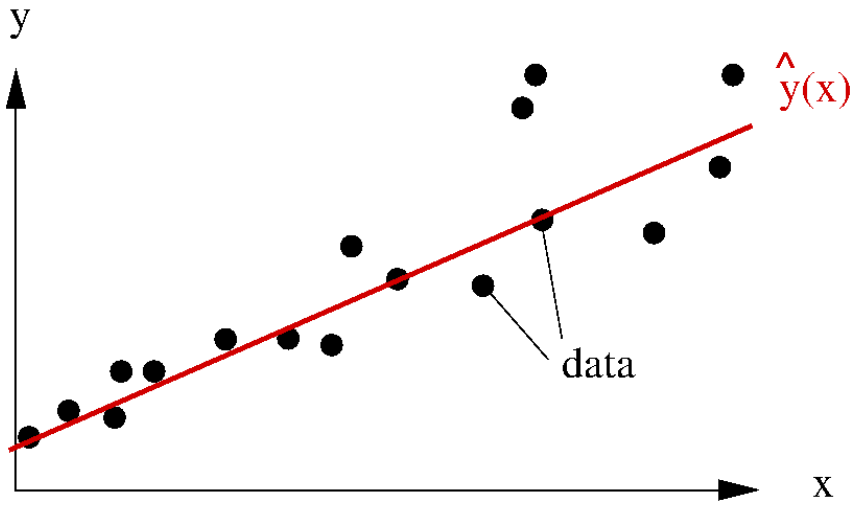

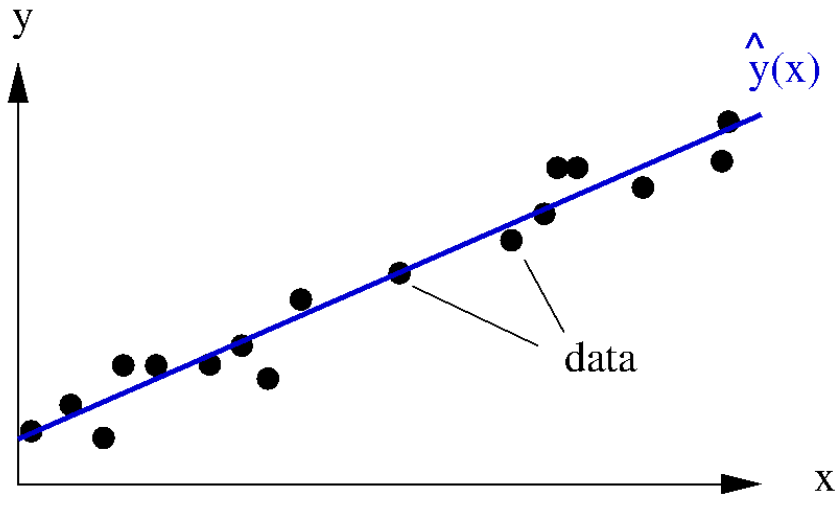

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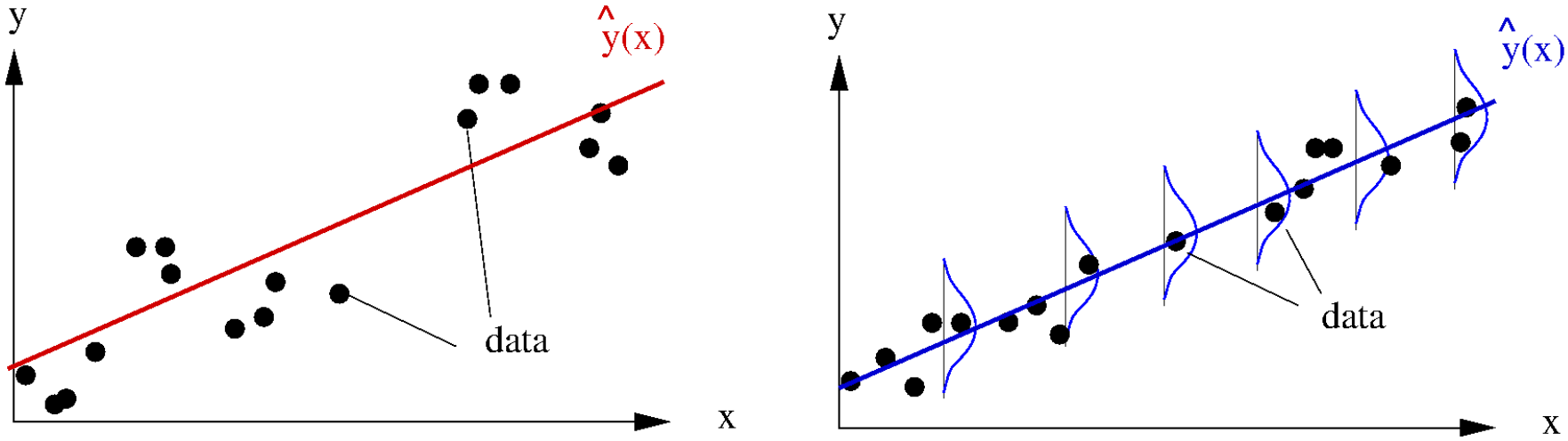

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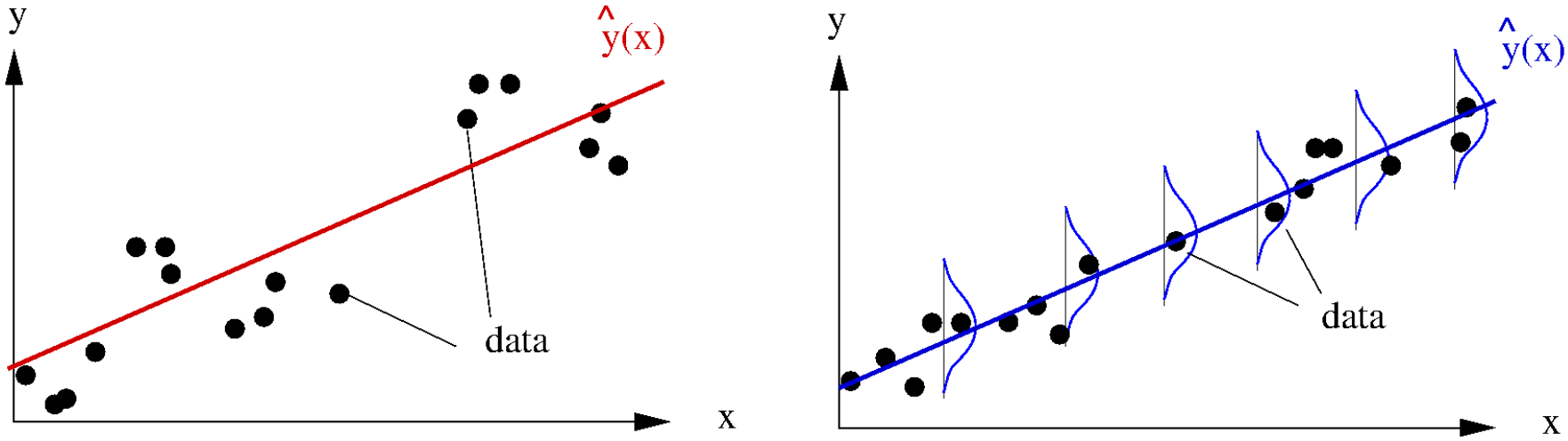

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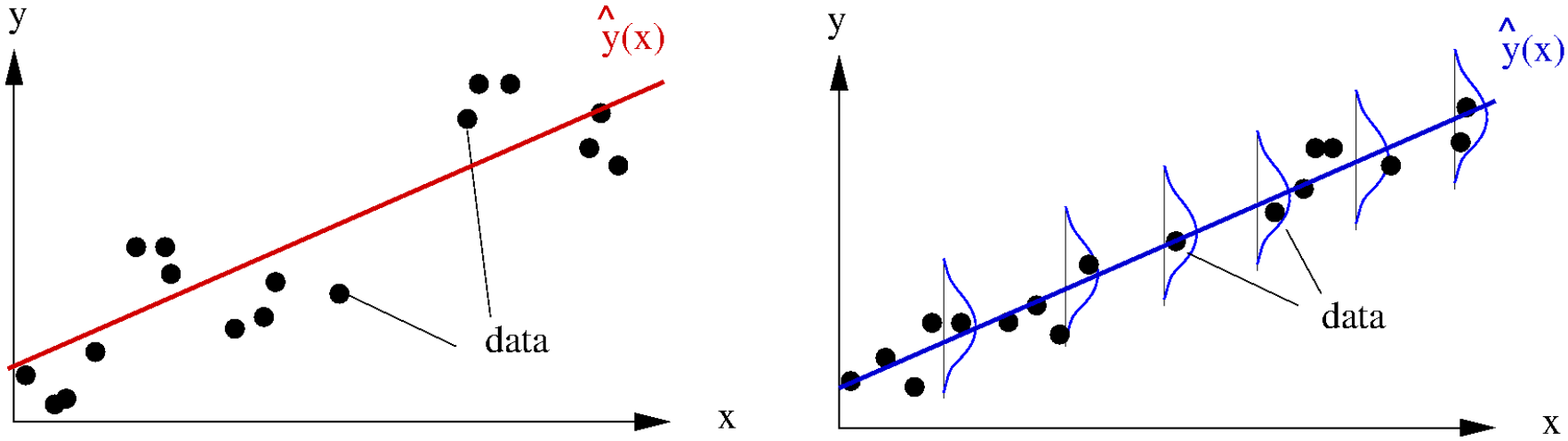

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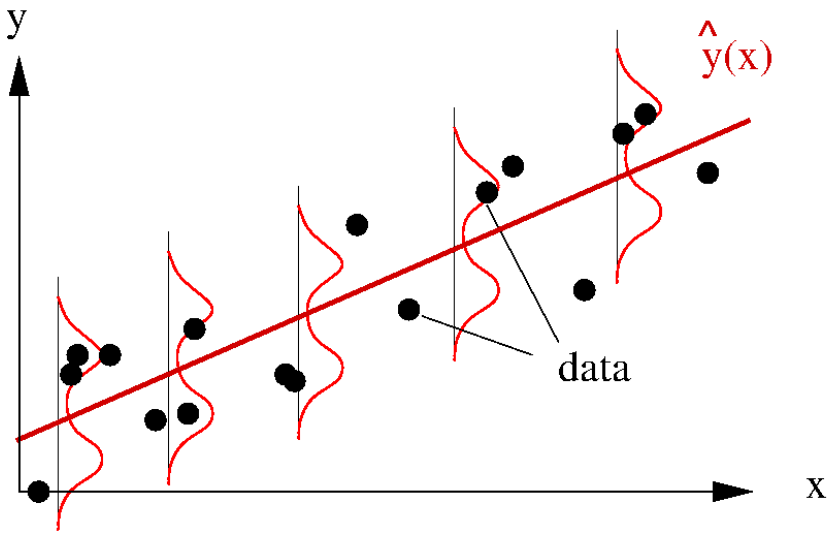

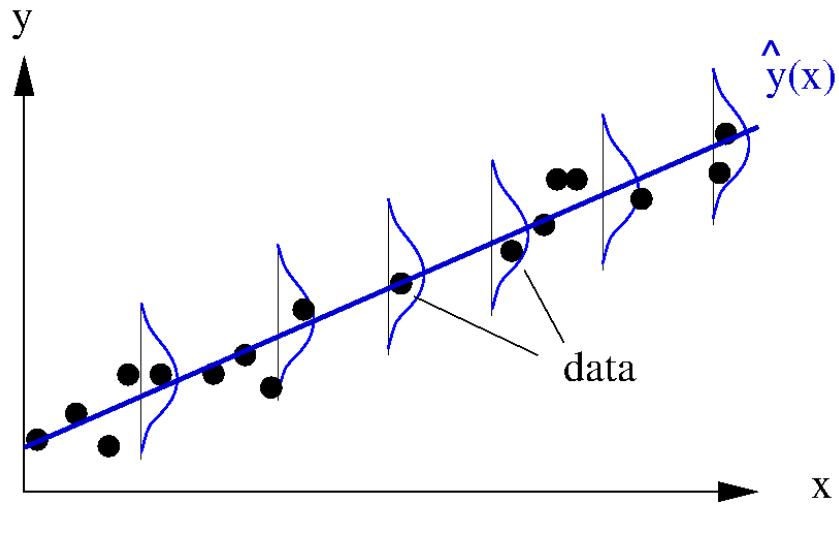

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$$\epsilon \sim \text{i.i.d.} N(0, \sigma^2) \quad \text{i.i.d.: identical independent distributions}$$

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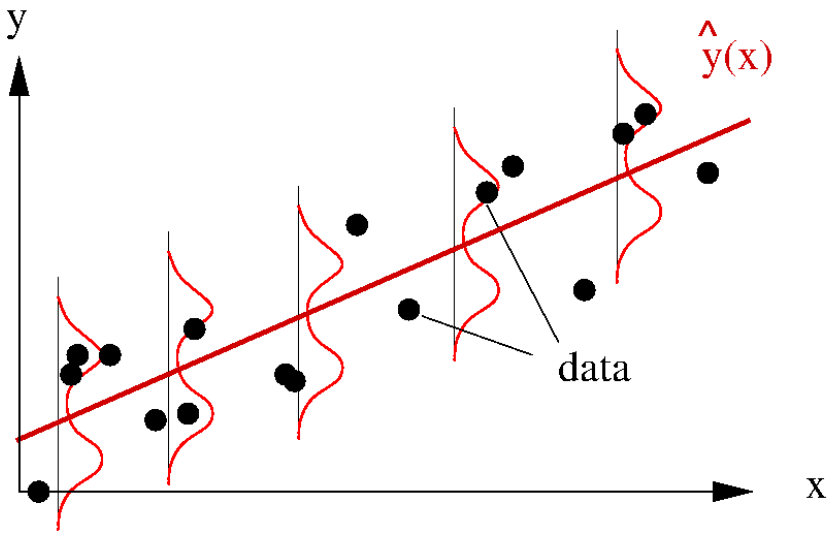

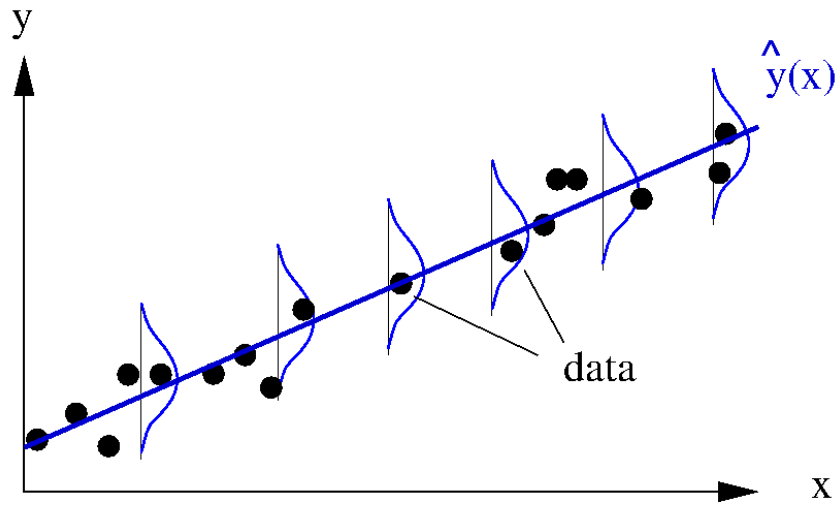

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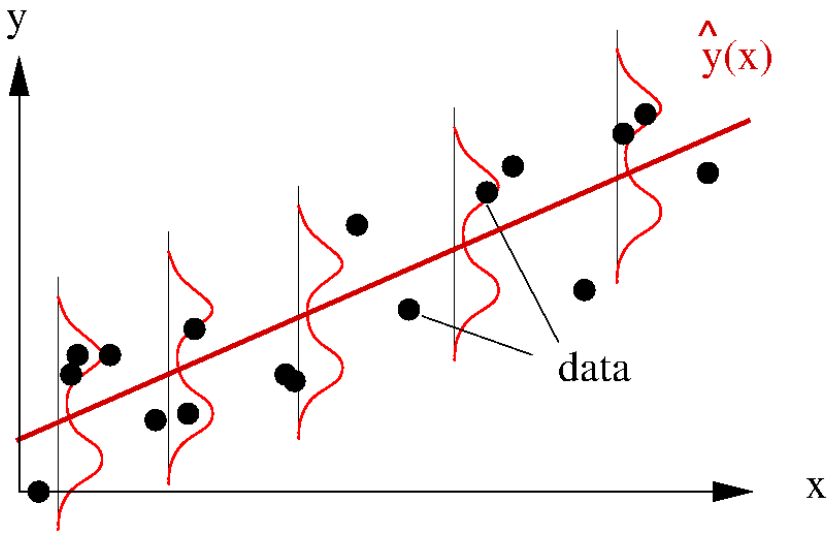

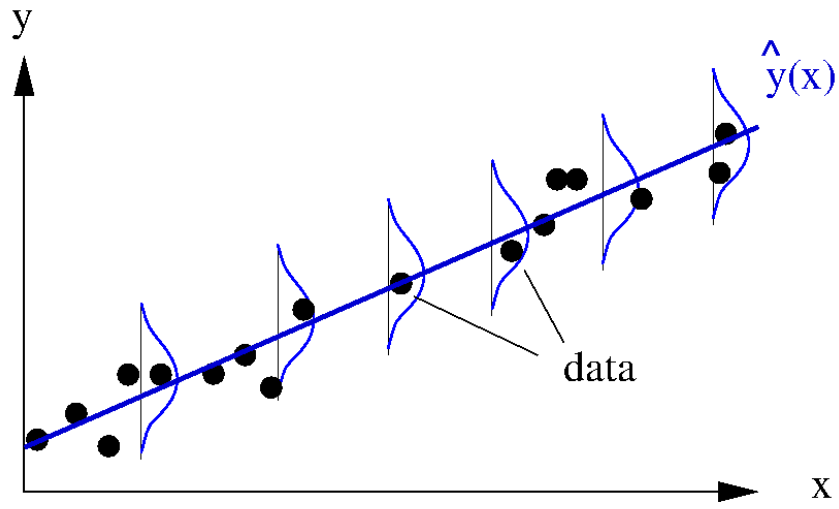

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- There must be more data sets (containing all exogenous variables and the endogenous variale, each) than model parameters: $n > p = J + 1$
- This means, the data should overdetermine the model which is the basis for fitting.
- **Consequence of a violation:** If $n = p$, the data determine the model exactly, i.e., it can be calibrated to zero residuals $\epsilon_i = 0$: *overfitting*. This is still **harmless** since OLS will detect it for you (zero residuals) and the inferential analysis will return a "0/0 error"
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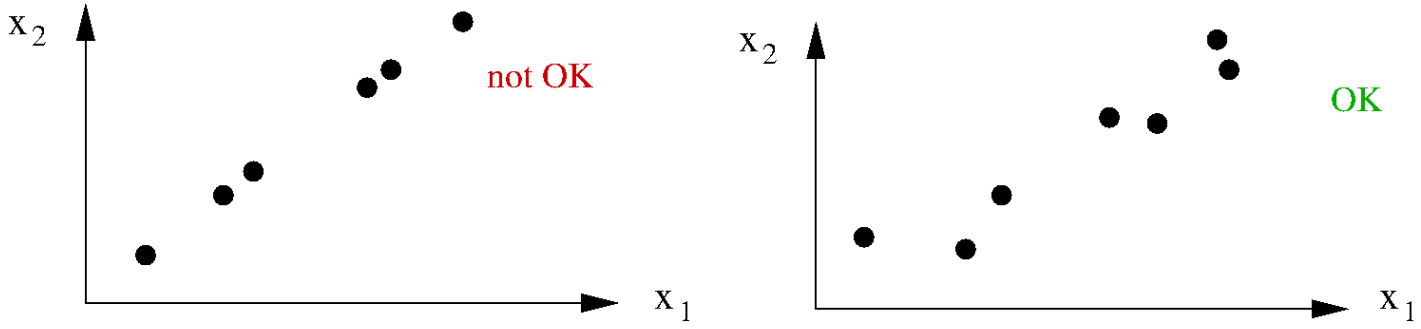

- A given exogenous variable must not be represented as a linear combination of other exogenous variables. Otherwise, the data matrix is *singular*
- However, nonperfect correlations $\neq \pm 1$ are allowed.
- Nonperfect correlations appear regularly, e.g., price vs quality
- **Consequences:** OLS **detects a perfect multicollinearity for you** by a "division by zero" error. A nearly perfect multicollinearity will lead to **large estimation errors**

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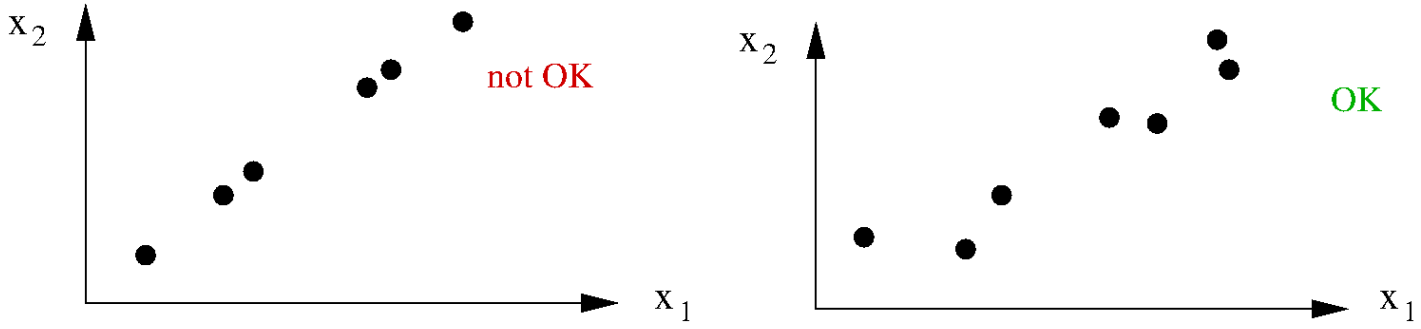

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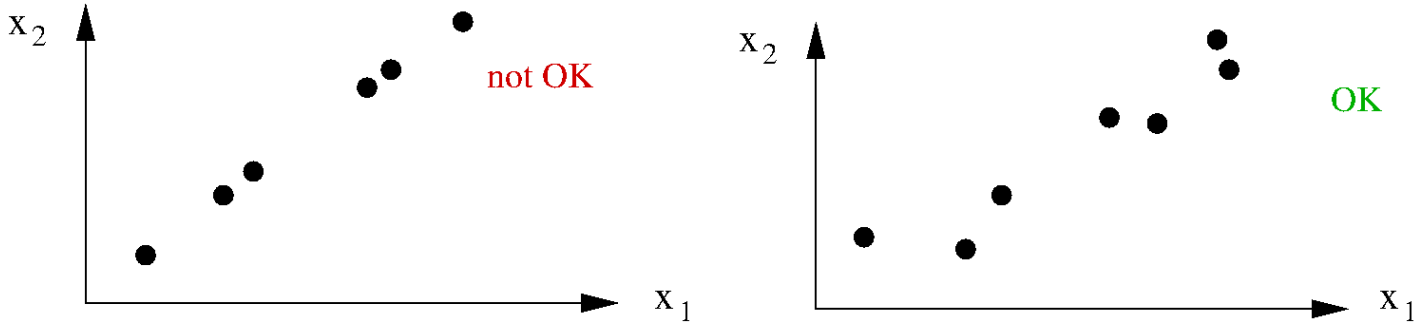

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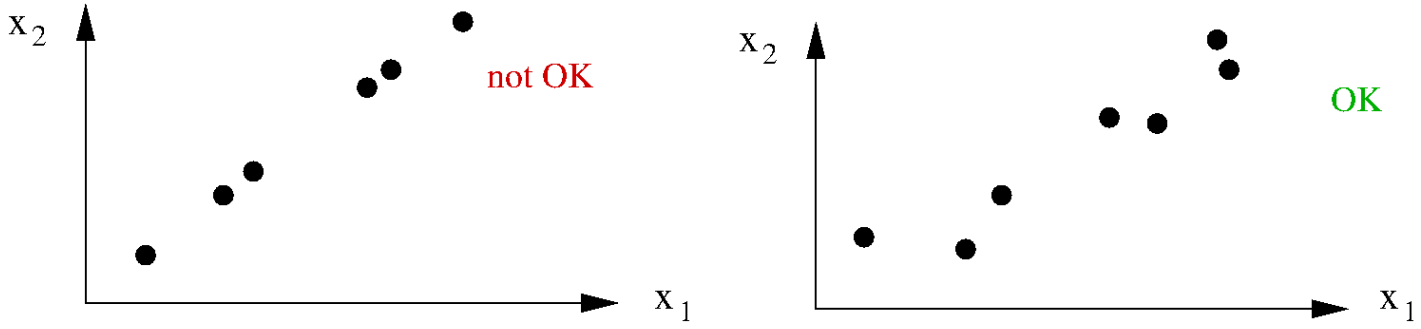

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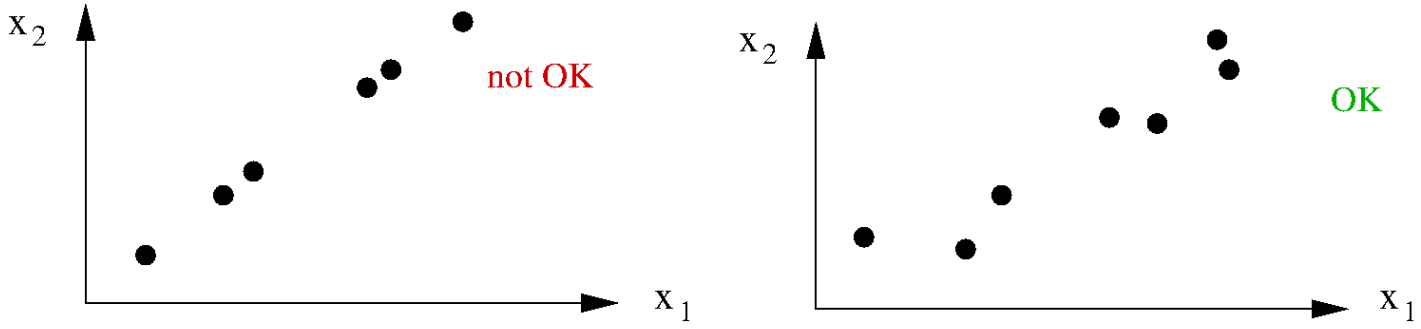

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How to detect multicollinearity

- Assume n data sets $\{x_{i0}, ..., x_{ij}, ..., x_{iJ}\}$, $i = 1, ..., n$ (the data sets also contain the endogenous variable but it is not relevant here)
- x_{ij} is the j^{th} exogenous factor in the i^{th} data set
- Multicollinearity exists if there is one exogenous factor x_k that can be expressed as a linear combination of all other factors $j \neq k$ in *all* data sets:

$$x_{ik} = \sum_{j \neq k} c_j x_{ij} \quad \forall i = 1, ..., n, \quad \text{constant } c_j$$

- A linear relation $x_2 = c_0 x_1$ is easy to detect but this is not the case for more complex relationships
- Solution: Check whether the **descriptive variance-covariance matrix**

$$S_{jk} = \frac{1}{n} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)$$

has the full rank $p = J + 1$, i.e., $\det \mathbf{S} \neq 0$

- For $n < p$, this is not satisfied trivially

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- Multicollinearity exists if there is one exogenous factor x_k that can be expressed as a linear combination of all other factors $j \neq k$ in *all* data sets:

$$x_{ik} = \sum_{j \neq k} c_j x_{ij} \quad \forall i = 1, ..., n, \quad \text{constant } c_j$$

- A linear relation $x_2 = c_0 x_1$ is easy to detect but this is not the case for more complex relationships
- Solution: Check whether the **descriptive variance-covariance matrix**

$$S_{jk} = \frac{1}{n} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)$$

has the full rank $p = J + 1$, i.e., $\det \mathbf{S} \neq 0$

- For $n < p$, this is not satisfied trivially

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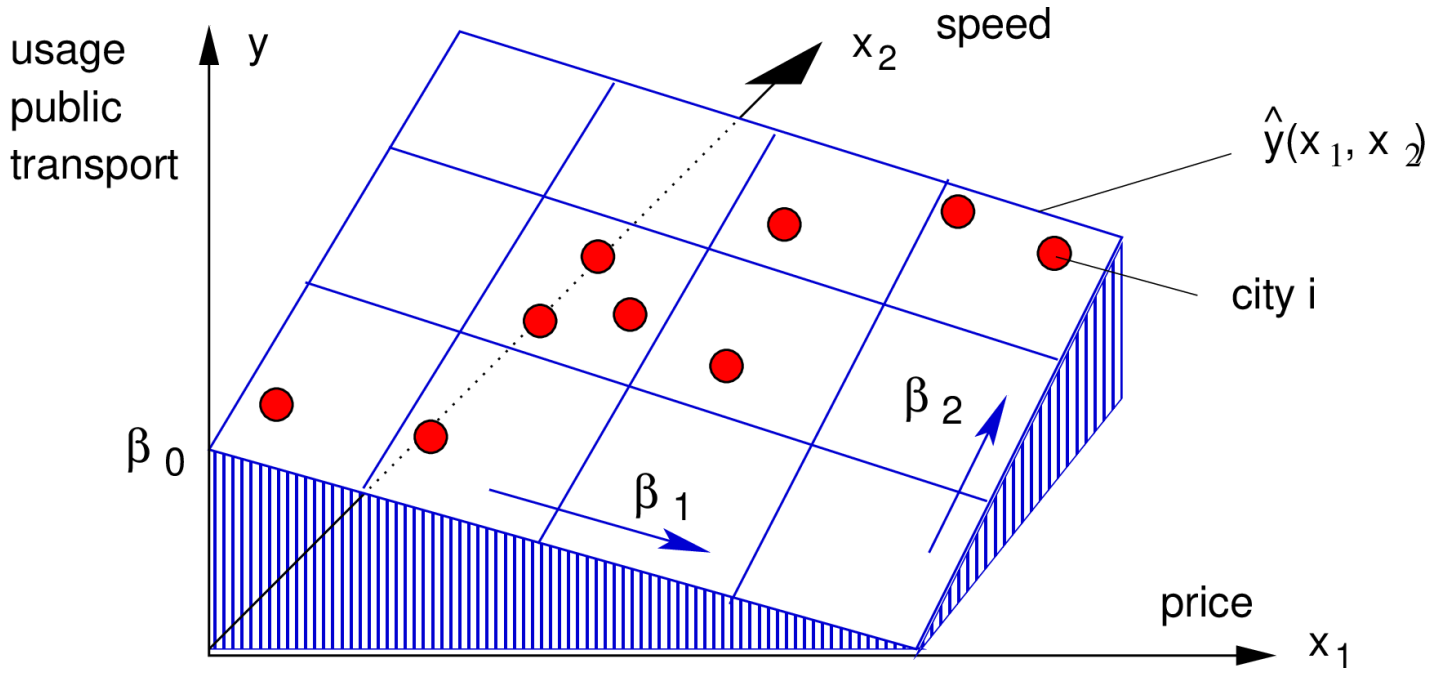
Data specification 2: example



- The normalized demand y_i for public transport in city i depends on the price x_{i1} and the quality x_{i2} (proxy: speed) of the service.
- Parameters: intercept β_0, price sensitivity β_1, appraisal for quality β_2.
- Price and quality are correlated but not perfectly so.
- This model structure is quite generic for products and services.

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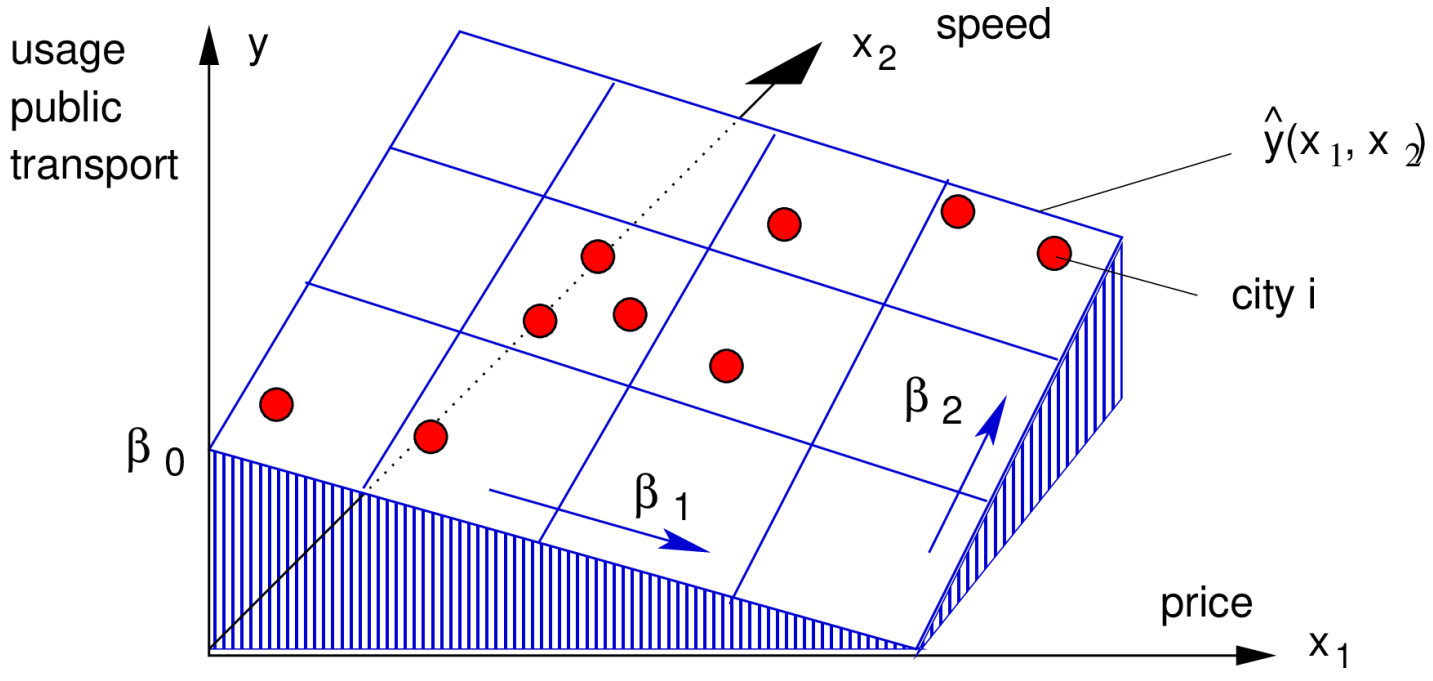

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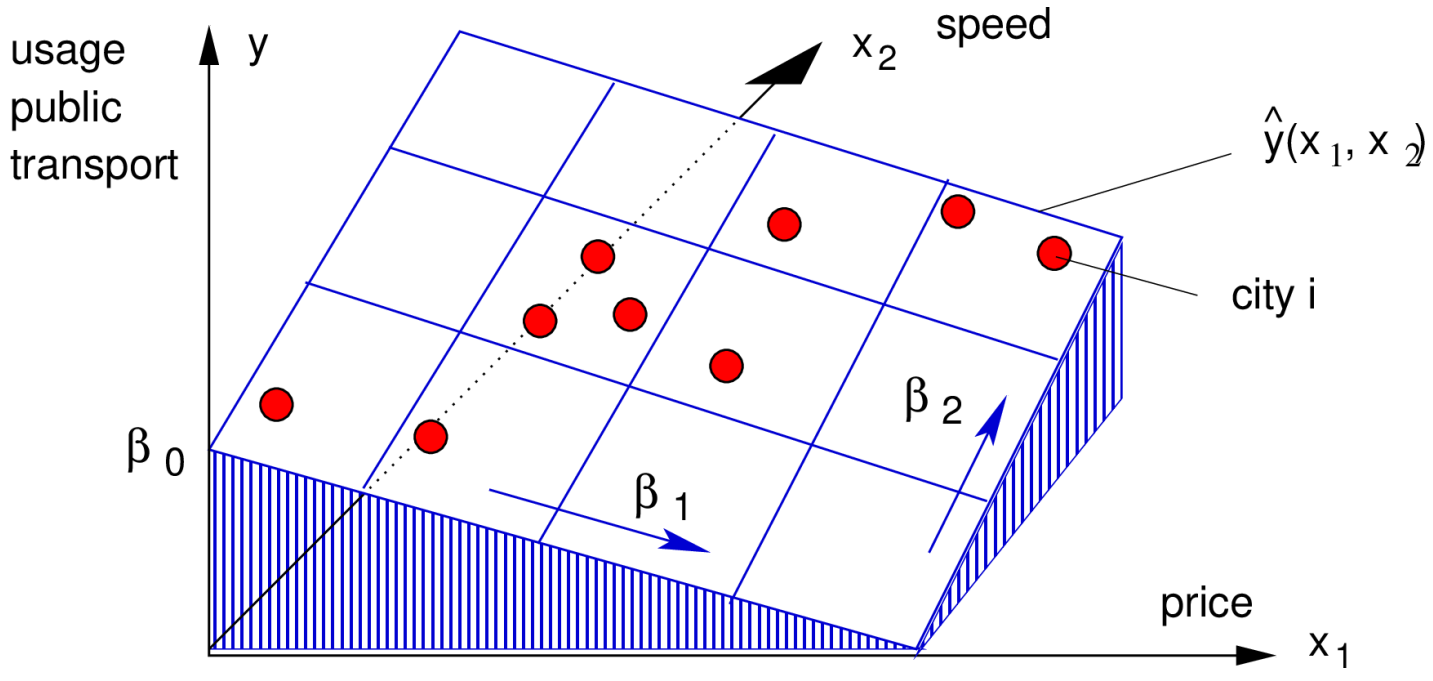

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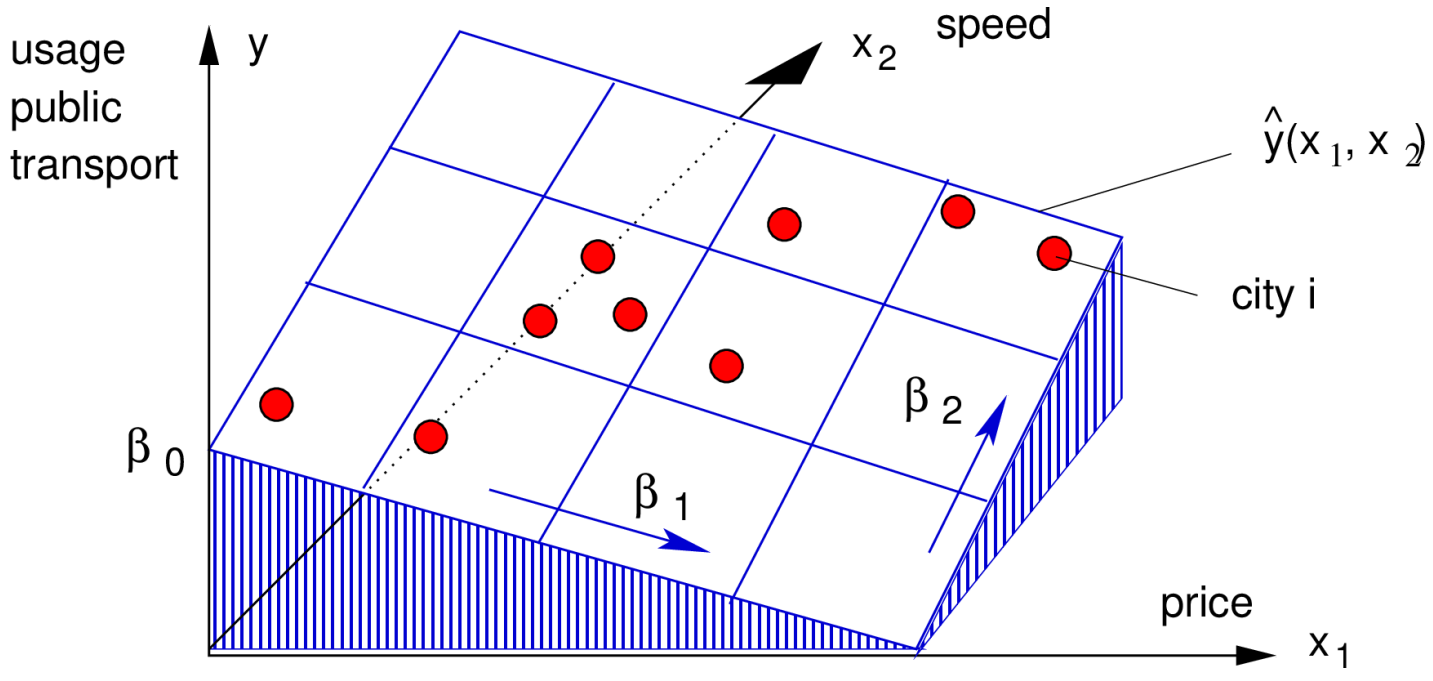

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2.3. Ordinary Least Squares (OLS) Estimation

- Given is a linear model of the form

$$y(\boldsymbol{x}) = \boldsymbol{\beta}'\boldsymbol{x} + \epsilon = \hat{y}(\boldsymbol{x}) + \epsilon, \quad \epsilon \sim i.i.d.\, N(0, \sigma^2)$$

  satisfying the Gauß-Markow specifications (the Gaussian distribution of the ϵ_i is not required, here)
- Given is also data in the form of n multidimensional data points containing all observations and satisfying the Gauß-Markow specifications as well:

$$\{\boldsymbol{p}_i = (x_{i0}, ..., x_{iJ}, y_i)', \quad i = 1, ..., n\}$$

- Searched for is a parameter estimator $\hat{\boldsymbol{\beta}}$ minimizing the sum of squared errors between data and model prediction with respect to the parameters:

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} S(\boldsymbol{\beta})$$

  where

$$S(\boldsymbol{\beta}) = \boldsymbol{\epsilon}'\boldsymbol{\epsilon} = (\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta})'(\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta}).$$

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Determining the OLS estimator

$$S = (\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta})' (\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta})$$

Determining the OLS estimator

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S &= (\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta})' (\boldsymbol{y} - \mathbf{X}\boldsymbol{\beta}) \\
\text{[distributivity } \rightarrow\text{]} &= \boldsymbol{y}'\boldsymbol{y} - (\mathbf{X}\boldsymbol{\beta})'\boldsymbol{y} - \boldsymbol{y}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})'\mathbf{X}\boldsymbol{\beta}
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Taking the derivative $\frac{\partial}{\partial \boldsymbol{\beta}}$ respecting $\frac{\partial}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}'\boldsymbol{a}) = \boldsymbol{a}$ and $\frac{\partial}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}'\mathbf{A}\,\boldsymbol{\beta}) = (\mathbf{A} + \mathbf{A}')\boldsymbol{\beta}$ with $\mathbf{A} = \mathbf{X}'\mathbf{X}$ symmetric:

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