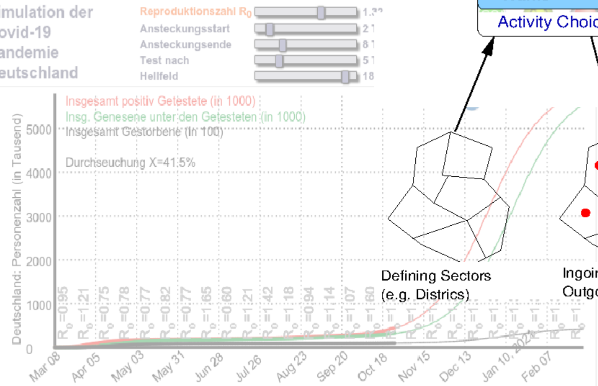
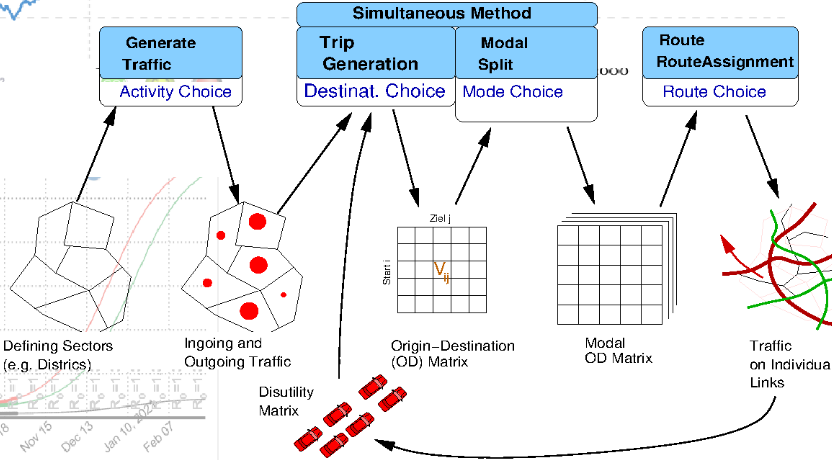
Traffic Econometrics Master's Course

Lecture 01: General





1.1 Scope of econometrics – from a mathematical point of view

Economic Theories

Mathematical Methods

Data, Statistics

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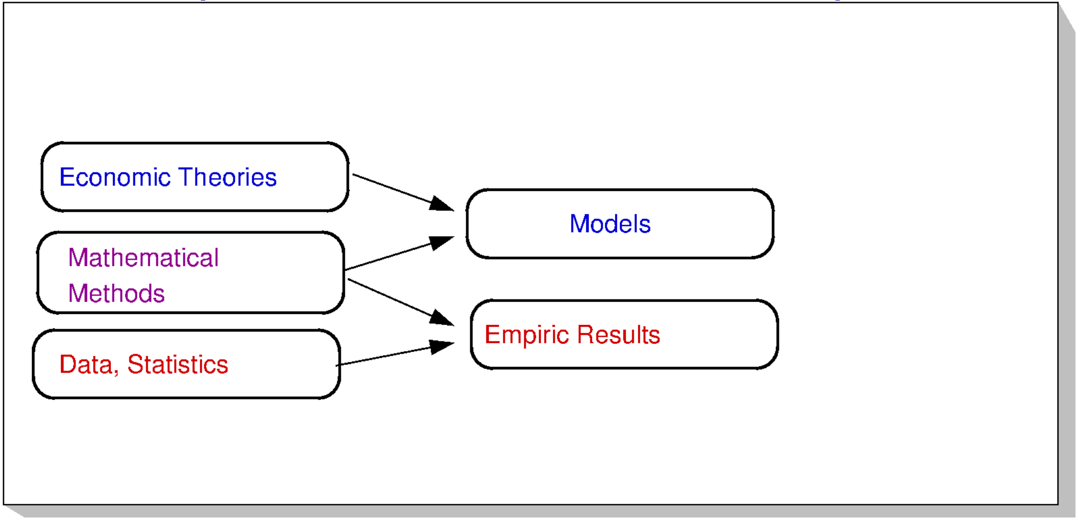

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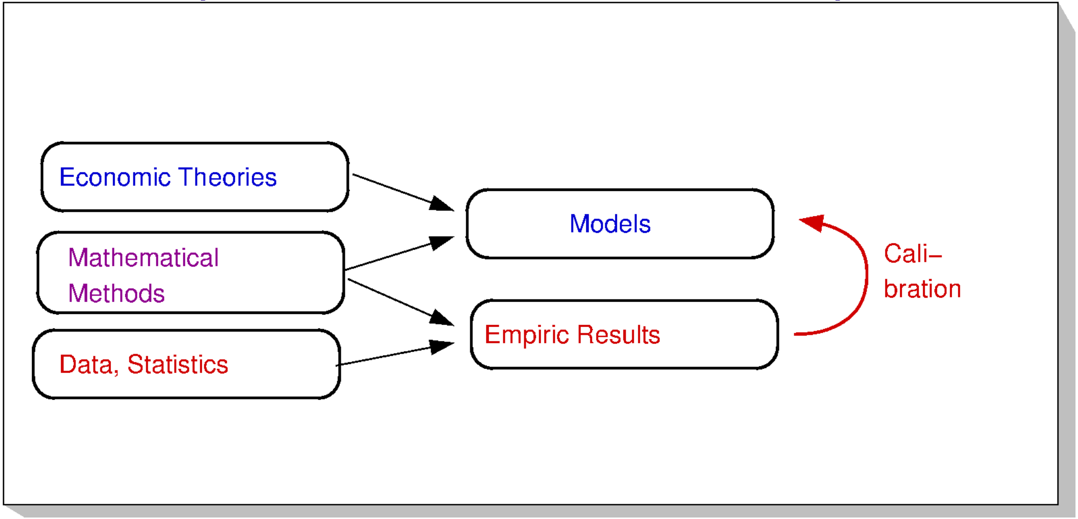

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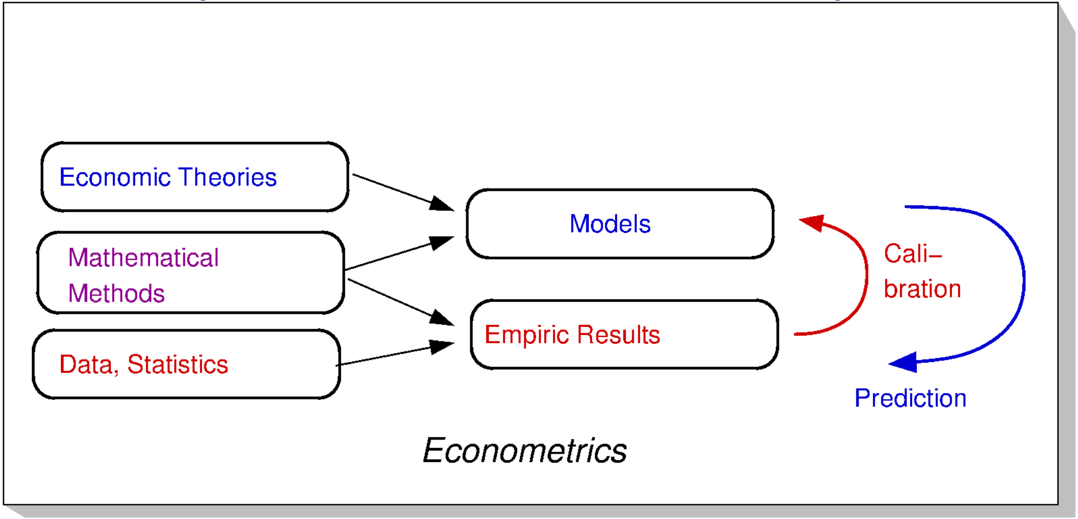

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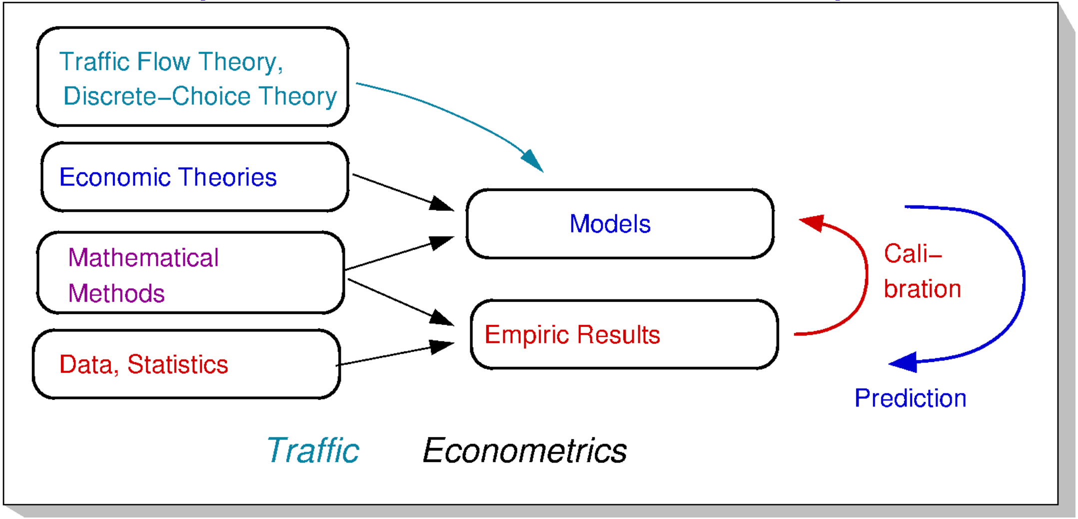

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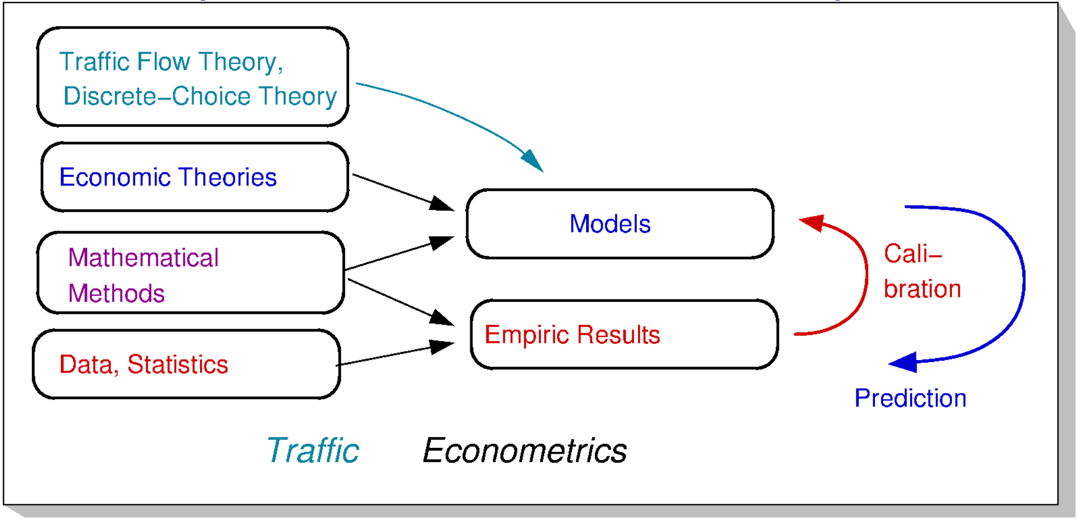
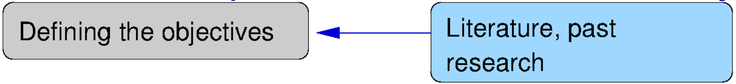


The field of **Traffic Econometrics** includes all mathematical models and statistical procedures to quantitatively analyze empirical (transportation) data with respect to economic effects.

1.2 General procedure of an econometric analysis



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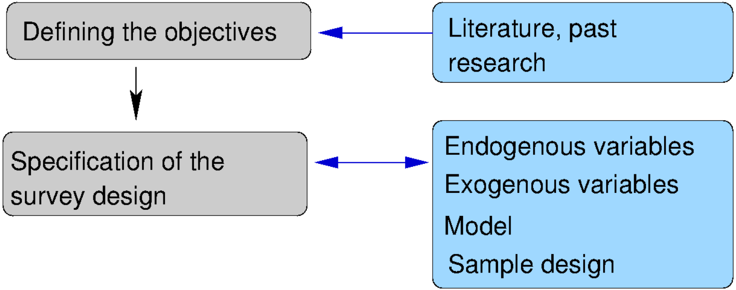

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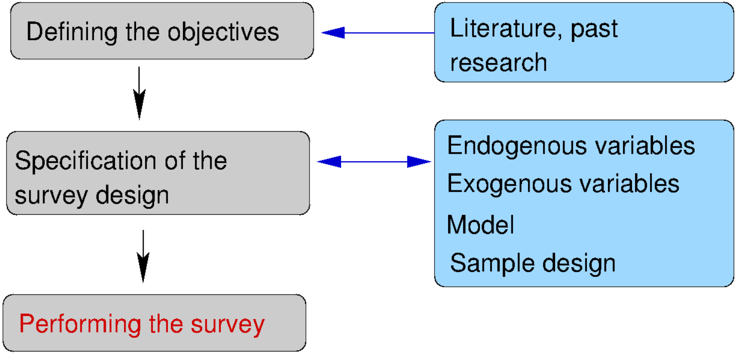

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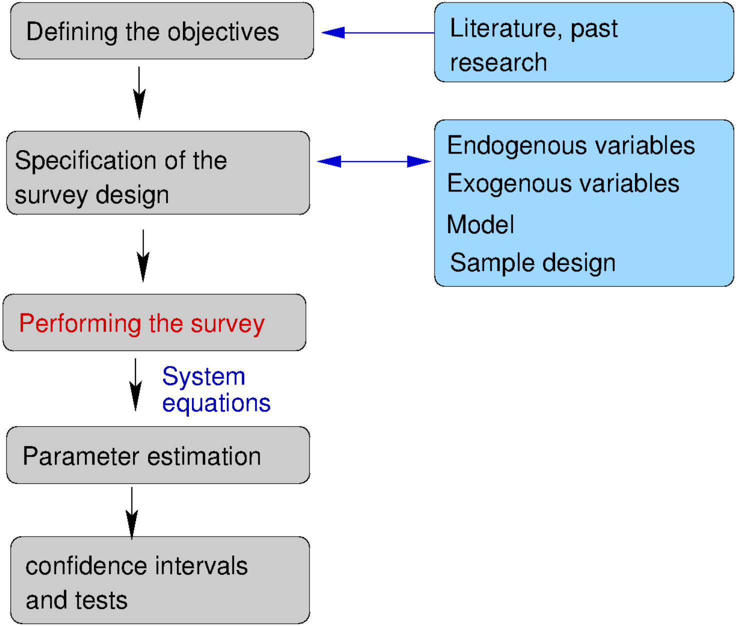

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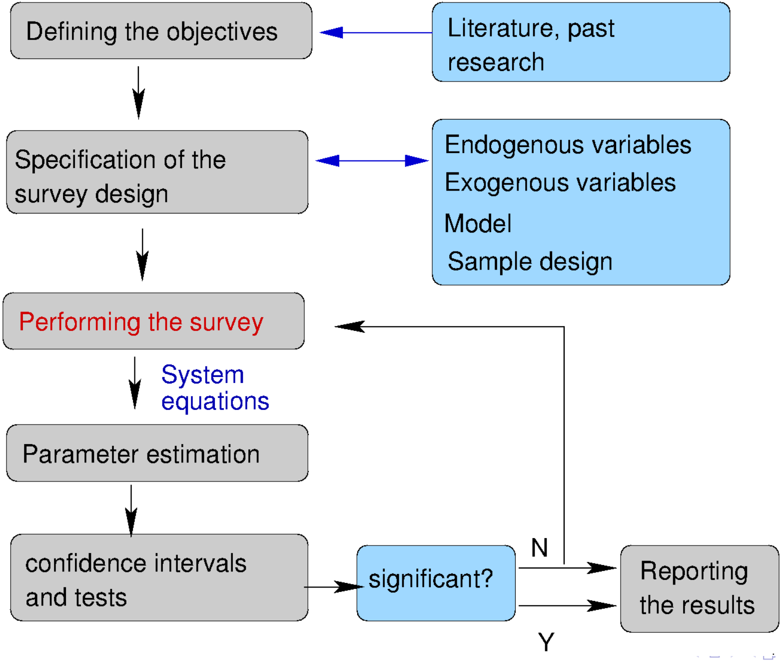
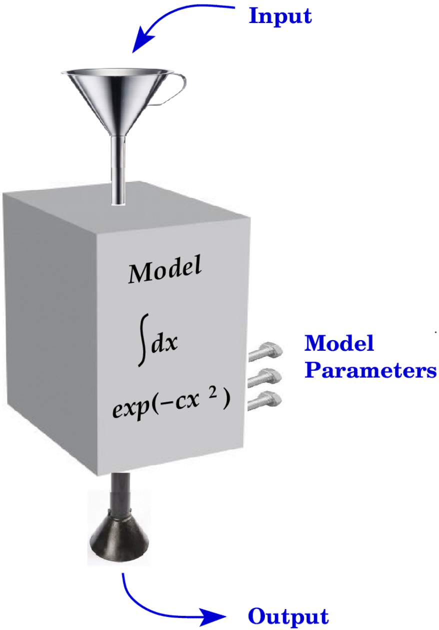


1.3 Information flow of an econometric model



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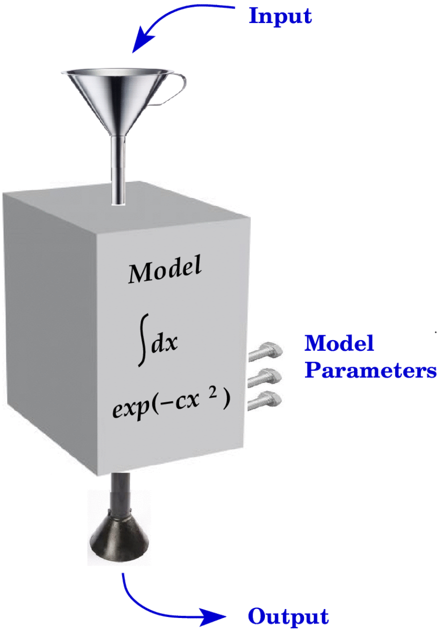

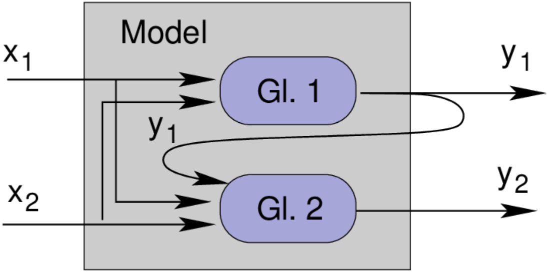

1.4 General Criteria for the Model Selection

Since econometrics describes things quantitatively, its basic language is mathematics and its basic concept a **(mathematical) model**

- The model must be compatible to the data and the research objective:
  - Model for fuel/energy consumption → real-valued output → e.g., **regression models**
  - Model for trip or mode choice → discrete output → e.g., **discrete-choice models**
  - Classification of different days with respect to the traffic demand profile (mid-workdays, sundays, holidays, ...) → econometric **discriminant analysis**

- If the model is a **discrete-choice model**, the questions of the survey must be sets of alternatives that are ...
  - *exclusive*: at most one alternative can be ticked
  - *complete*: at least one alternative may be ticked ⇒ exactly one
  - sufficiently *different*.

  ? How to formulate a question regarding the kind of schools visited?
  ? For route choice, we have two routes that only differ in that one route contains a small detour of to go to a bakery. Which criterion is violated?

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1.5 Econometric models – a Closer Look

$$Y_k = f_k(\tilde{x}_1, ..., \tilde{x}_m, .., \tilde{x}_M, \beta_0, ..., \beta_j, ..., \beta_J) + \epsilon_k = f_k(\tilde{\boldsymbol{x}}, \boldsymbol{\beta}) + \epsilon_k$$

- A model consists of one or more equations $Y_k = f_k(\tilde{x}, \beta) + \epsilon_k$ explaining the output quantity Y_k as a function of the input $\tilde{x}$
- to *tune* the model, there are **model parameters** β
- a model can be stochastic ($\epsilon_k \neq 0$) or deterministic ($\epsilon_k = 0$)
- the above formulation is the most general one and includes *all* conceivable econometric models.

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1.5.1 Endogenous Variables

- The name **endogenous variable** comes from the Greek *endo* ($\epsilon\nu\delta$) and the suffix *gen* ($\gamma\epsilon\nu o\varsigma$) meaning *generated from the inside [the model]*
- in systems theory, the endogenous variables are the **output**
- mathematically, the endogenous variable are the **dependent variables**
- logically, they are the **explained variables**

Example mode choice: Y_k: #decisions for mode k (e.g., walking, cycling, public transport, car, combined/others)

Formulate conditions for the Y_k in order to ensure that the choice set is exclusive and complete

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- an additive fixed function of explanatory variables is often called a **factor**

Since the exogenous variable describe explanatory factors, they are generally considered to be deterministic with all stochasticity deferred to additional random terms ϵ_k.

Example mode choice: $\tilde{x}_m$ can be travel times or costs for the different modes, and also socioeconomic attributes such as age, gender, income, or the possession of a car

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1.5.3 Random terms

The **random terms** ϵ_k, also called **residual terms** (from *residuum*: the rest) summarize all what is not known and cannot be explained by the exogenous/explanatory variables:

Scio nescio (I know that I do not know)

possible reasons for ϵ_k:

- model does not include all relevant exogenous variables **(watch out for bias!)**
- all relevant factors are there but are not bundled to appropriate linear factors (e.g., explaining the fuel consumption by a linear function of the speed)
- the data used for model calibration contain errors
- in case of human decisions:

man $\neq$ machine; homo $\neq$ *homo oeconomicus*

Example mode choice: we ignored the weather or the additional utility to stop by at a bakery (not provided by public transport), or need of a certain mode (car) for subsequent trips of that day

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Scio nescio (I know that I do not know)

possible reasons for ϵ_k:

- model does not include all relevant exogenous variables **(watch out for bias!)**
- all relevant factors are there but are not bundled to appropriate linear factors (e.g., explaining the fuel consumption by a linear function of the speed)
- the data used for model calibration contain errors
- in case of human decisions:

man $\neq$ machine; homo $\neq$ *homo oeconomicus*

Example mode choice: we ignored the weather or the additional utility to stop by at a bakery (not provided by public transport), or need of a certain mode (car) for subsequent trips of that day

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Model parameters

The **model parameters** β_j, $j = 0, \ldots, J$ *tune* the model to fit the data

- The parameters are determined by fitting the model to *learning data sets*, a process called **calibration**
- To test the *explanatory/prediction power* of a model, the calibrated model is applied to *test data sets* with known output, a process called **validation**
- In contrast to the exogenous variables changing from application to application, the parameters are *fixed* after calibration.

The existence of well validated models is the *raison d' être* for econometrics as such

Example mode choice: Parameters characterize, e.g., the monetary **value of time** (VoT) in €/h

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Model functions

The **model function** is a mathematical representation of the economic process under investigation. The model's mathematical structure must reflect reality as well as possible:

- linear *vs.* nonlinear
- deterministic *vs.* stochastic
- single or multiple equations that may be linked, chained, or with feedback
- which exogenous variables are relevant?

The model structure defines the qualitative aspects and the model parameters the quantitative aspects of whatever is to be investigated

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Mathematical structure I: linear vs. nonlinear

Four steps from linearity to nonlinearity:

1. **Truly linear models:**

$$Y = \hat{y}(\tilde{\boldsymbol{x}}, \boldsymbol{\beta}) + \epsilon = \sum_{j=0}^{J} \beta_j \tilde{x}_j + \epsilon = \boldsymbol{\beta}' \tilde{\boldsymbol{x}} + \epsilon$$

   Because of linearity, each endogenous variables has its own uncoupled single equation, so it is enough to consider a single component.

2. **Parameter-linear (quasi-linear) models**

$$Y = \hat{y}(\tilde{\boldsymbol{x}}, \boldsymbol{\beta}) + \epsilon = \sum_{j=0}^{J} \beta_j g_j(\tilde{\boldsymbol{x}}) + \epsilon = \sum_{j=0}^{J} \beta_j x_j + \epsilon = \boldsymbol{\beta}' \boldsymbol{x} + \epsilon$$

   If it is reasonable to assume fixed (generally nonlinear) functions $x_j = g_j(\tilde{\boldsymbol{x}})$, we have a linear model with the **factors** x_j becoming the "new" exogenous variables

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Example: average distance travelled in vehicles per year

- y: distance [km/year] (deterministic because of "average")
- $\tilde{x}_1$: income [€/year]
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! 0.15 % decrease in kilometrage per increase of the fuel costs by 1 %

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! $x_0 = 1$: constant; $x_1 = \tilde{x}_1$: increase with income ($\beta_1 > 0$); $x_2 = \tilde{x}_2$: price sensitivity ($\beta_2 < 0$); $x_3 = \tilde{x}_1\tilde{x}_2$: increase of price sensitivity (becoming less negative) with increasing income ($\beta_3 > 0$)

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Mathematical structure I: linear vs. nonlinear 3

3. **Nonlinear models that can be linearized**
   Classical example: unlimited growth

$$G(t) = G_0 e^{t/\tau + \epsilon}$$

- endogenous variable G: growth measure, e.g., company size, #items sold of a newly introduced product
- exogenous variable t: time
- parameter G_0: initial growth measure
- parameter τ: time for growing by a factor of $e = 2.71\ldots$
- random multiplicative factor e^{ϵ}

$$\textbf{Linearisation:} \quad Y(t) = \ln G(t) = \ln G_0 + \frac{t}{\tau} + \epsilon$$

Reformulation by setting $x_0 = 1$, $x_1 = t$, $\beta_0 = \ln G_0$, $\beta_1 = 1/\tau$:

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$$G(t) = G_0 e^{t/\tau + \epsilon}$$

- endogenous variable G: growth measure, e.g., company size, #items sold of a newly introduced product
- exogenous variable t: time
- parameter G_0: initial growth measure
- parameter τ: time for growing by a factor of $e = 2.71...$
- random multiplicative factor e^{ϵ}

Linearisation: $Y(t) = \ln G(t) = \ln G_0 + \frac{t}{\tau} + \epsilon$

Reformulation by setting $x_0 = 1$, $x_1 = t$, $\beta_0 = \ln G_0$, $\beta_1 = 1/\tau$:

Standard form: $Y(x) = \beta' x + \epsilon$

Mathematical structure I: linear vs. nonlinear 3

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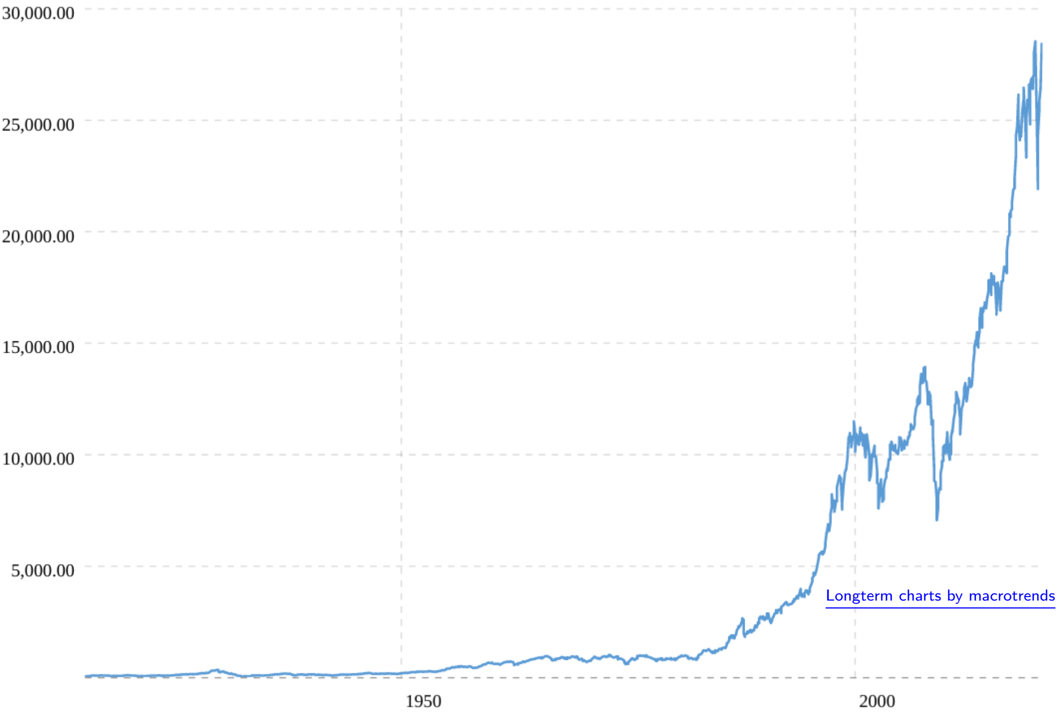

Longterm charts by macrotrends

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Mathematical structure I: linear vs. nonlinear 4

4. **Irreducibly nonlinear models** Classical example: limited growth

$$y(t) = \frac{y_s}{1 + (y_s/y_0 - 1)e^{-t/\tau}}$$

- Solution of the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y(t)}{\tau}\left(1 - \frac{y(t)}{y_s}\right)$ for the initial value $y(t_0) = y_0$
- Plot for $t_0 = 1950$, $y_0 = 3\,\%$, $y_s = 60\,\%$, and $\tau = 10\,\mathrm{years}$

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! For example, the penetration rate for passenger cars per person.

Mathematical structure I: linear vs. nonlinear 4

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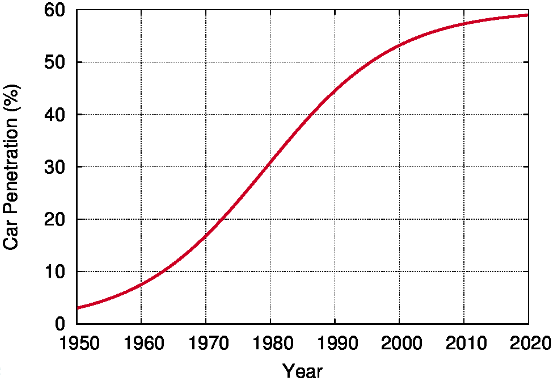

Mathematical structure I: linear vs. nonlinear 4

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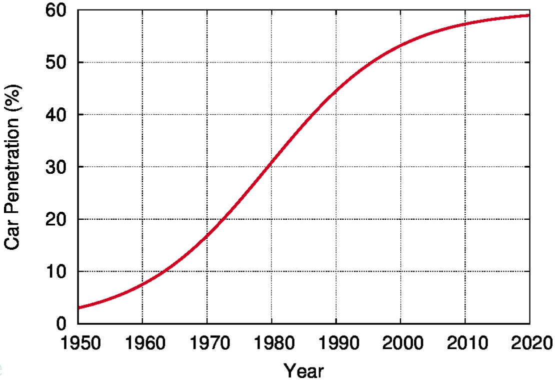

Mathematical structure I: linear vs. nonlinear 4

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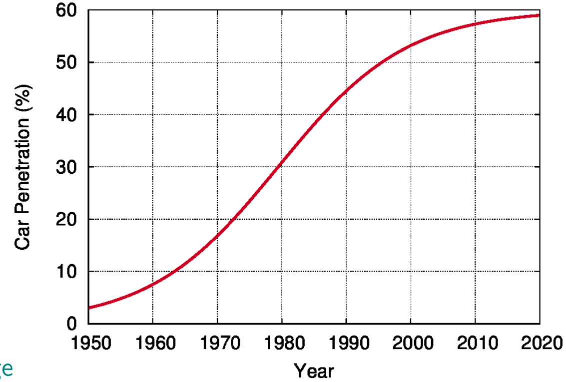
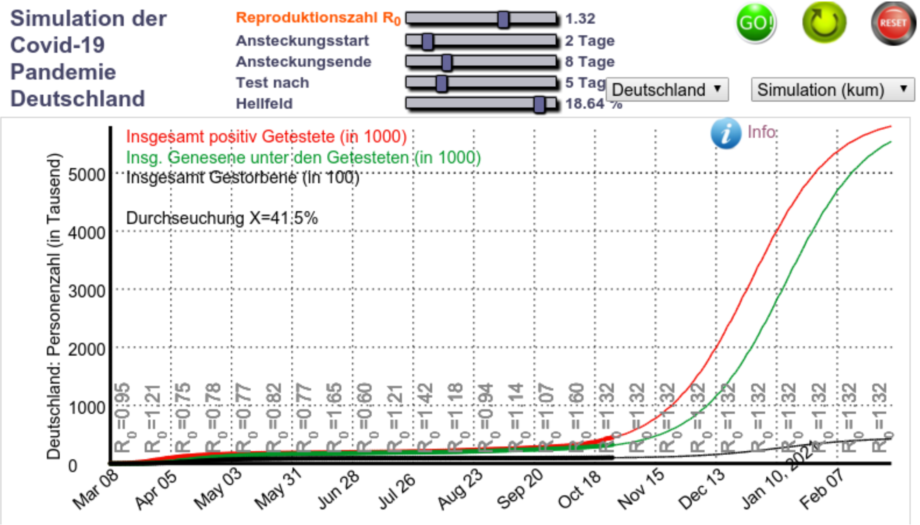


Actual example: Corona simulation



Simplest macroscopic SI (susceptible-infected) model:

$y(t+\tau) = y(t) + R_0 y(t)(1-y(t))$ $\rightarrow$ same model as above $\Rightarrow$ Lecture 01a

Mathematical structure II: Other criteria

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- #exogenous variables: 1: **univariate**; ≥ 2 **multivariate** models
- #endogenous variables: single- *vs.* multi-equation models
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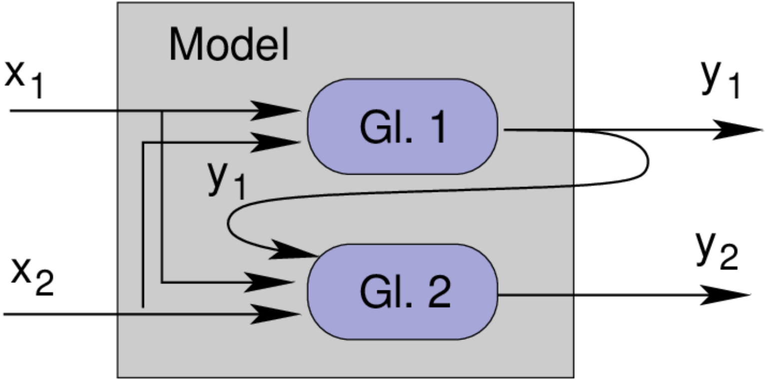
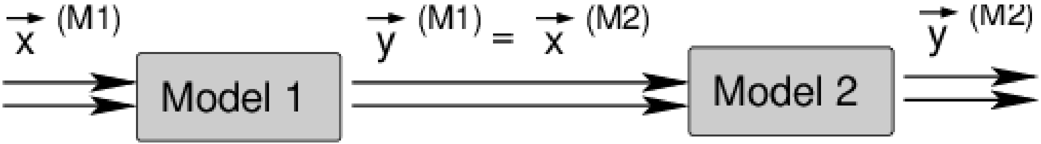
Linking



One or more endogenous variables of a model equation serve as exogenous variables of other model equations

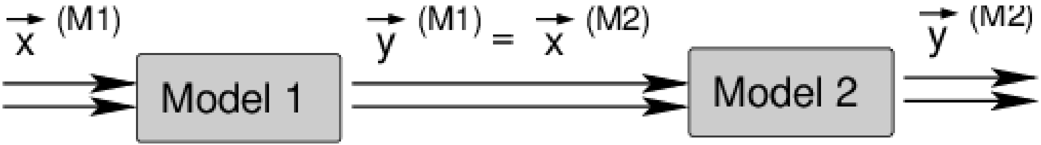
Chaining



The endogenous variables of Model 1 serve as exogenous variables of Model 2

- Special case of chaining: **time evolution**: the endogenous variables at time t are the exogenous variables at the next time step $t + \Delta t$
- The model itself is generally the same in all steps (autonomous model)

Chaining



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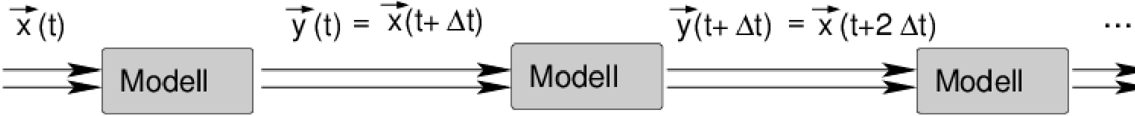

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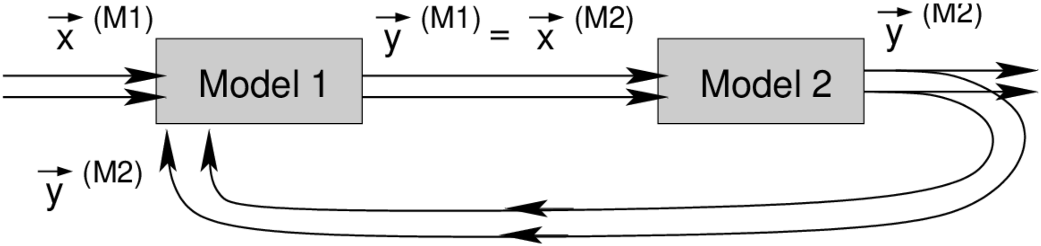
Feedback



Combination of chaining and linking

Complex example: Four-Step Model of transportation planning