Exam to the Lecture
Traffic Dynamics and Simulation
SS 2025

Solutions

Total 120 points

Problem 1 (20 points)

(a) A single counting detector can be used for determining flows, e.g., for a daily demand curvve.However, this was none of the mentioned use cases.

Data type	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Single counting detector						
Single loop detector				✓		
Two single counting detectors at a distance	✓		✓			
Double-loop detector	✓			✓		
Floating-car data	✓	✓				✓
Trajectory data	✓	✓		✓	✓	✓

Comments: Due to the unknown penetration rate, FC data cannot be used to estimate extensive quantities like density or flow. However, they can be used for intensive quantities such as speed. Use case (iii) needs counting detectors at a distance to the traffic light and at the traffic light such that an algorithmus is able to estimate the number of incoming vehicles and shortens/extends the phase accordingly.

(b) (i) Eulerian: Fixed frame of reference. What a stationary detector would see. Typical frame of reference for macroscopic models such as LWR.

(ii) Lagrangian: Moving frame of reference from a driver's viewpoint. Used for most micromodels.

(c) Advective acceleration from the driver's point of view:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = 0 + 10\,\mathrm{m/s} * 0.1\,\mathrm{s}^{-1} = 1\,\mathrm{m/s}^2$$

Problem 2 (50 points)

Given is following acceleration equation for a car-following model as a function of the gap s, the speed v, and the leader's speed v_l:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f = \min\left[a\left(1-\frac{v}{v_0}\right),\ a\left(1-\frac{s^*(v,v_l)}{s}\right)\right], \quad s^*(v,v_l) = s_0 + vT + \frac{v(v-v_l)}{2\sqrt{ab}}$$

(a)
- Input: gap s, the speed v, and the leader's speed v_l
- Output: The acceleration $f = \frac{\mathrm{d}v}{\mathrm{d}t}$
- Parameters: As the IDM: Desired speed v_0, following time gap T, minimum gap s_0, maximum accleeration a, desired/comfortable deceleration b

(b) The acceleration function is continuous but not smooth. Because of the minimum function, it has non-differentiable parts ("kinks")

(c) See (a). The parameterisation is suitable for city traffic because 15 m/s correspond to 54 km/h.

(d) Fundamental diagram means a homogeneous steady state of identical drivers and vehicles. Homogeneous means $\frac{\partial}{\partial x} = 0$ or $v_l = v$. Steady state means $\frac{\partial}{\partial t} = 0$ With $\frac{\partial}{\partial x} = 0$, this means also $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x} = 0$.

(e) In steady state, the gap variable s^* is given by

$$s^*(v) = s_0 + vT$$

- Free traffic: Then, the first expression determines the minimum and we have $v = v_0$. This applies if $1 - s^*/s = 1 - (S_0 + vT)/s > 0$ or $s > s_0 + vT$
- Congested traffic: Then, the second expression is the minimum and equal to zero. This means, $v < v_0$ (otherwise, the first expression is zero and determines the minimum). Then, we have just $s_e(v) = s*(v) = s_0 + vT$ or, inverting this, $v_e(s) = (s - s_0)/T$

Together, we have

$$v_e(s) = \begin{cases} v_0 & s \geq s_0 + v_0 T \\ \frac{s-s_0}{T} & s_0 \leq s < s_0 + v_0 T \end{cases}$$

Note: For $s < s_0$, this model would drive backwards which needs to be suppressed.

(f)
- Free part: Simply $Q(\rho) = V_0\rho$.
- Congested part: With the definition of the density ρ, number of vehicles per distance, we have the relation

$$\rho = \frac{1}{l_{\text{veh}} + s)} \overset{\text{cong.}}{=} \frac{1}{l_{\text{veh}} + s_0 + VT)}$$

or

$$V(\rho) = \frac{1}{T}\left(\frac{1}{\rho} - l_{\text{eff}}\right), \quad l_{\text{eff}} = l_{\text{veh}} + s_0, Q = \rho V(\rho)$$

Together, we have

$$Q(\rho) = \begin{cases} V_0\rho & \rho \leq \frac{1}{l_{\text{eff}} + v_0 T} \\ (1 - \rho l_{\text{eff}}) & \text{otherwise,} \end{cases}$$

i.e., a tridiagonal fundamental diagram.

- Density at capacity $\rho_c = (l_{\text{eff}} + v_0 T)^{-1} = (40\,\text{m})^{-1} = 25\,/\text{km}$
- capacity $Q_{\max} = v_0 \rho_c = 0.375\,/\text{s}$
- wave speed (not required) $w = -l_{\text{eff}}/T = -5\,\text{m/s}$

(g) At $v = v_0$ and with a red traffic lights, i.e., the stopping line corrresponds to a stopped leader, we have the acceleration

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(s, v, v) = \begin{cases} 0 & s > s_{\text{br}} = s_0 + v_0 T + \frac{v_0^2}{2\sqrt{ab}} \\ a\left(1 - \frac{s_{\text{stop}}}{s}\right) & \text{otherwise.} \end{cases}$$

This means, the deceleration begins if $s < s_{\text{br}}$ and it increases smoothly since $f \to 0$ if $s \to s_{\text{stop}}$.

- The critical distance s_{br} where the deceleration begins corresponds to the minimum gap s_0, the time gap $v_0 T$, and the kinematic braking distance $v_0^2/(2b')$ for $b' = \sqrt{ab}$ being the geometrical mean of a and b
- Numerical value $s_{\text{br}} = 89.25\,\text{m}$

(h) Partial derivative of f with respect to v and v_l at steady, f_v and f_l, respectively: First, one needs to distinguish the two cases. At the steady state, we have either $v = v_0$ (first expression is relevant) or $s = s_e^* = s_e = s_0 + vT$ (second expresion relevant, $v < v_0$ or $s < s_0 + v_0 T$. In the first case, we simply have

$$(f_v)_{\text{free}} = -\frac{a}{v_0}, \quad (f_l)_{\text{free}} = 0.$$

In the second interacting case, we have

$$\begin{aligned}
(f_v)_{\text{int}} &= -\frac{a}{s}\frac{\partial s^*(v, v_l)}{\partial v}\bigg|_e \\
&= -\frac{a}{s}\left(T + \frac{2v - v_l}{2\sqrt{ab}}\right)\bigg|_e \\
&= -\frac{a}{s_e}\left(T + \frac{2v_e - v_e}{2\sqrt{ab}}\right) \\
&= -\frac{a}{s_e}\left(T + \frac{v_e}{2\sqrt{ab}}\right) \\
&= -\frac{2aT + \sqrt{\frac{a}{b}}v_e}{2s_e}.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
(f_l)_{\text{int}} &= -\frac{a}{s}\frac{\partial s^*(v, v_l)}{\partial v_l}\bigg|_e \\
&= +\frac{a}{s}\,\frac{v}{2\sqrt{ab}}\bigg|_e \\
&= \sqrt{\frac{a}{b}}\,\frac{v}{2s_e}.
\end{aligned}$$

For all steady states, we have

$$f_v < 0, \quad f_l \geq 0$$

satisfying the two plausibility criteria for the sensitivities to speed and leading speed.

(i) Here, we have to distinguish between the free and interacting regimes as well.

- Free, $s_e \geq s_c = s_0 + v_0 T$: We have $v_e'(s) = 0$ and therefore the string stability criterium

$$0 < -\frac{f_v}{2} = \frac{a}{2v_0}$$

which obviously is always satisfied: Free traffic is unconditionally string stable.

- Interacting, $s_e < s_c = s_0 + v_0 T$: Here, $v_e'(s) = \frac{\partial}{\partial s}((s - s_0)/T) = 1/T$, so the string stability condition reads

$$\begin{aligned} v_e'(s) &\leq \frac{1}{2}(f_l - f_v) \\ \frac{1}{T} &\leq \frac{aT + \sqrt{\frac{a}{b}} v_e}{s_e} \\ \frac{1}{T} &\leq \frac{aT + \sqrt{\frac{a}{b}} v_e}{s_0 + v_e T} \\ \frac{s_0 + v_e T}{T} &\leq aT + \sqrt{\frac{a}{b}} v_e \end{aligned}$$

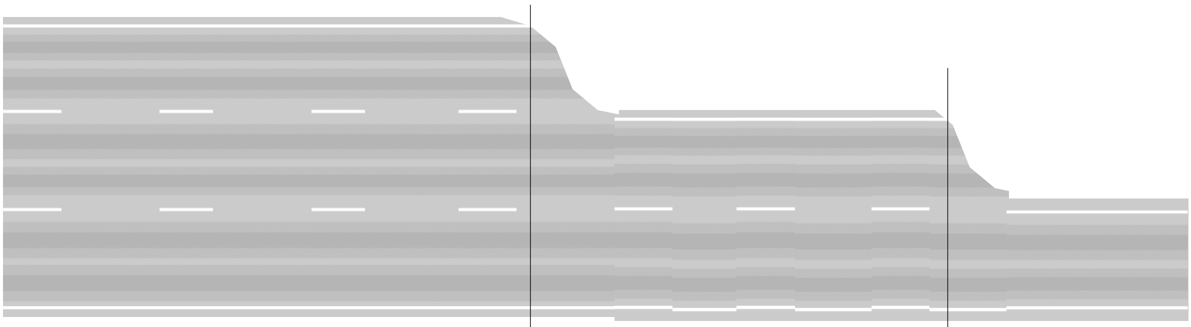
Solving this for v_e gives

$$v_e \left(1 - \sqrt{\frac{a}{b}}\right) \leq \frac{aT^2 - s_0}{T}$$

Generally, this leads to several cases *The relevant first case already gives full marks!*

  * If $aT^2 - s_0 \geq 0$ and $a = b$ (as here), string stability is always satisfied
  * If $aT^2 - s_0 \geq 0$ and $a > b$, we have string stability for $v_e > (aT^2 - s_0)/(-T(\sqrt{a/b} - 1))$ which is always satisfied (watch the reversal of the inequality operator if both sides are multiplied with a negative factor!)
  * If $aT^2 - s_0 \geq 0$ and $a < b$, we have string stability for $v_e \leq (aT^2 - s_0)/(T(1 - \sqrt{a/b}))$ (and of course for $v_e = v_0$)
  * If $aT^2 - s_0 < 0$ and $a \leq b$, we have never string stability for $v < v_0$
  * If $aT^2 - s_0 < 0$ and $a > b$, we have string stability for $v_e \geq (aT^2 - s_0)/(T(1 - \sqrt{a/b}))$

Problem 3 (50 points)
Given is a two-stage lane drop from three to two to one lane in a city traffic context (no traffic lights) according to following sketch



Traffic flow is modeled with an LWR model with a triangular fundamental diagram with the parameters

$$V_0 = 15\,\mathrm{m/s}, \quad T = 1.5\,\mathrm{s}. \quad l_{\text{eff}} = \frac{1}{\rho_{\max}} = 7.5\,\mathrm{m}.$$

(a) Capacity per lane:

$$Q_{\max} = \frac{v_0}{v_0 T + l_{\text{eff}}} = 0.5\,\mathrm{s}^{-1} = 1\,800\,/\mathrm{h}$$

(b) With $Q_{\text{in}} = 1\,500\,\mathrm{veh/h}$, even the single-lane section is able to supply the demand, so no traffic breakdown happens. In all sections, the total flow and densities and the speeds are given by

$$\begin{aligned} Q_I^{\text{tot}} &= Q_2^{\text{tot}} = Q_3^{\text{tot}} = Q_{\text{in}} = 1\,500\,\mathrm{veh/h}, \\ V_I &= V_2 = V_3 = V_0 = 15\,\mathrm{m/s} = 54\,\mathrm{km/h} \\ \rho_I^{\text{tot}} &= \rho_2^{\text{tot}} = \rho_3^{\text{tot}} = \frac{Q_{\text{in}}}{V_0} = 27.7\,\mathrm{veh/km} \end{aligned}$$

For the per-lane intensive quantities Q and ρ, one has to divide by the number of lanes, so

$$Q_I = \frac{Q_I^{\text{tot}}}{3}, \quad Q_2 = \frac{Q_I^{\text{tot}}}{2}, \quad \rho_I = \frac{\rho_I^{\text{tot}}}{3}, \quad \rho_2 = \frac{\rho_I^{\text{tot}}}{2},$$

(c) The local capacities are

$$C_I = 3Q_{\max} = 5\,400\,\mathrm{veh/h}, \quad C_2 = 2Q_{\max} = 3\,600\,\mathrm{veh/h}, \quad C_3 = Q_{\max} = 1\,800\,\mathrm{veh/h},$$

so, Sections I and II can accomodate the new demand and a breakdown happens at the beginning of Section III at $x_{\text{breakd}} = 1.8\,\mathrm{km}$ In free traffic, the sudden flow increase information propagates at the free-flow speed $c_{ab} = V_0$ (everything happens at the left side of the triangular fundamental diagram), so

$$t_{\text{breakd}} = \frac{x_{\text{breakd}}}{V_0} = 120\,\mathrm{s}$$

(d) Section IIa is the upstream free-flow regime with the per-lane state variables dictaed by the inflow (information propagates downstream in free flow), so

$$\begin{aligned} Q_{2\mathrm{a}} &= \frac{Q_{\mathrm{in}}}{2} = 1\,200\,\mathrm{veh/h} = \frac{1}{3}\,\mathrm{veh/s}, \\ \rho_{2\mathrm{a}} &= \rho_{\mathrm{free}}(Q_{2\mathrm{a}}) = \frac{Q_{2\mathrm{a}}}{V_0} = 22.2\,\mathrm{veh/km}. \end{aligned}$$

Section IIb is dictated by the bottleneck capacity (information flows upstrema in the congested states, so

$$\begin{aligned} Q_{2\mathrm{b}} &= \frac{C_{\mathrm{bottl}}}{2} = \frac{Q_{\mathrm{max}}}{2} = 0.25\,\mathrm{veh/s} = 900\,\mathrm{veh/h}, \\ \rho_{2\mathrm{b}} &= \rho_{\mathrm{cong}}(Q_{2\mathrm{b}}) = \rho_{\mathrm{max}}(1 - Q_{2\mathrm{b}}T) = 83.3\,\mathrm{veh/km}. \end{aligned}$$

The propagation velocity of the transition IIa $\rightarrow$ IIb is given by the shockwave formula:

$$c_{2\mathrm{a}2\mathrm{b}} = \frac{Q_{2\mathrm{b}} - Q_{2\mathrm{a}}}{\rho_{2\mathrm{b}} - \rho_{2\mathrm{a}}} = -1.36\,\mathrm{m/s} = -4.91\,\mathrm{km/h}$$

(e) The congested region just propagates over the lane drop, so there is no longer a Section IIa (i.e., the complete Section II is congested) and a new Section Ib (part of Section I is congested).

(f) The information propagates upstream throughout all congested regions, so the flow information is ultimately still given by $Q_{\mathrm{Ib}}^{\mathrm{tot}} = C_{\mathrm{bottl}}$. Likewise, the free information propagates downstream, so the free flow is given by $Q_{\mathrm{Ia}}^{\mathrm{tot}} = Q_{\mathrm{in}}$. Thus $Q_{1\mathrm{a}} = Q_{\mathrm{in}}/3 = 800\,\mathrm{veh/h}$ and $Q_{1\mathrm{b}} = C_{\mathrm{bootl}}/3 = 600\,\mathrm{veh/h}$. For the shockwave velocity, we need again the correpsonding densities:

$$\rho_{\mathrm{Ia}} = \frac{Q_{\mathrm{Ia}}}{V_0} = 0.0148\,\mathrm{veh/m}, \quad \rho_{\mathrm{Ib}} = \rho_{\mathrm{cong}}(1 - Q_{\mathrm{Ib}}) = 0.1\,\mathrm{veh/m}$$

Hence

$$c_{\mathrm{IaIb}} = \frac{Q_{\mathrm{Ib}} - Q_{\mathrm{Ia}}}{\rho_{\mathrm{Ib}} - \rho_{\mathrm{Ia}}} = -0.65\,\mathrm{m/s} = -2.54\,\mathrm{km/h}$$

Remarkable *(not required)*, the congested speed in Section I, $V_{1b} = Q_{1\mathrm{b}}/\rho_{1\mathrm{b}} = 1.67\,\mathrm{m/s}$ is nearly half of that in the congested Section II, $V_{2\mathrm{b}} = Q_{2\mathrm{b}}/\rho_{2\mathrm{b}} = 3\,\mathrm{m/s}$

(g) Fundamental diagram of total flows and densities: