Lecture 07: Macroscopic Second-Order Models

7.1 General Mathematical Form

In contrast to the LWR models, **second-order models** have their own dynamic equation for the dynamic speed. They come in two forms: **local** and **nonlocal**.

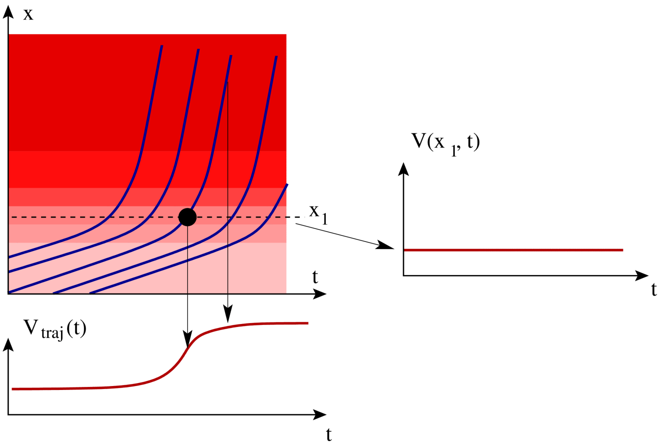
$$\frac{\mathrm{d}V(x,t)}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)V + \frac{1}{\rho}\frac{\partial P(\rho)}{\partial x} = A[\rho, V]$$ **local formulation**

- $\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)V(x,t)$ is the acceleration from the driver's point of view (Lagrangian formulation)
- The "traffic pressure" $P(\rho)$ is a statistical effect caused by speed variations
- The acceleration functional describes the aggregated vehicle accelerations:

$$A[\rho(x,t), V(x,t)] = f_{\text{loc}}\left(\rho, V, \frac{\partial\rho}{\partial x}, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \ldots\right)$$

- the derivatives of the pressure and acceleration terms are crucial since, without them, this model class would be *unconditionally unstable*

Acceleration in the Lagrangian view

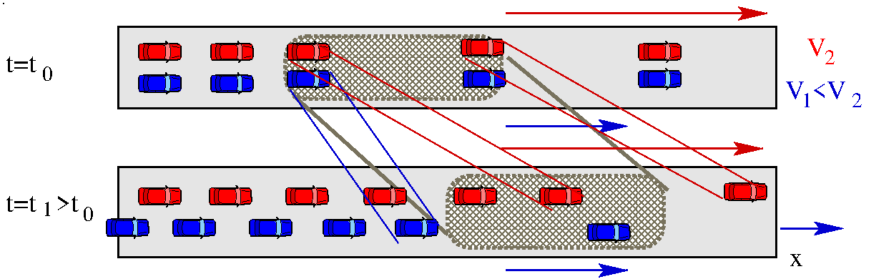


Derive the expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$ by looking at the speed change

$$\mathrm{d}V = \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{\partial V}{\partial x}\,\mathrm{d}x = \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}\right)\mathrm{d}t$$

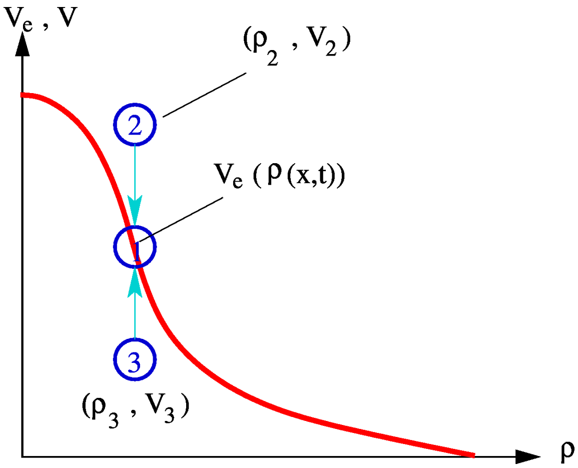
1. changes of the speed field $\frac{\partial V}{\partial t}$ at a fixed location,
2. changes of the speed field $V\frac{\partial V}{\partial x}$ when moving along the spatially varying field

The “pressure term”: a purely statistical effect



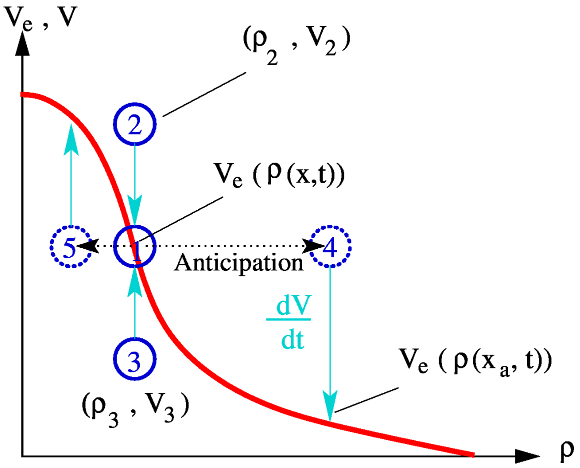
- Neither the red nor the blue vehicles accelerate but the red vehicles are twice as fast as the blue ones all the time
- Macroscopically, the local density and speed in the hatched region of length Δx is relevant. At $t = 0$, we have $V(t = 0) = \frac{V_1+V_2}{2}$
- While being advected at speed V (advection term!), the faster (slower) cars enter the hatched area from the upstream (downstream) end.
- Due to the density gradient, more faster vehicles entering than leaving the region, less slower vehicles entering than leaving $\Rightarrow$ macroscopic local speed changes if there is both finite speed variance Θ and a density gradient (here $V(t_1) = (2V_2 + V_1)/3 > V(0)$)

True acceleration I: relaxation



The relaxation term $f_{\text{relax}} = (V_e(\rho) - V)/\tau$ realizes a desire of the drivers to "come back" to the fundamental diagram in the relaxation time τ

True acceleration II: anticipation



The anticipation terms $f_{\text{antic}} = \gamma_1 \dfrac{\partial \rho}{\partial x} + \gamma_2 \dfrac{\partial V}{\partial x}$ anticipate the situation at some forward location Give the expression when anticipating the relaxation process at Point I at a distance $1/\rho$ $f_{\text{relax}} + f_{\text{antic}} = (V_e(\rho_a) - V)/\tau$ where $V_e(\rho_a) = V_e(\rho) + V_e'(\rho)\frac{\partial \rho}{\partial x}\frac{1}{\rho}$

True acceleration III: diffusion

- The formation mechanism of shock waves/fronts (see last lecture) is hardly suppressed by the anticipation mechanism
- However, in second-order models, shock waves have unfavourable numeric properties
- Therefore, an ad-hoc term $f_{\text{diffus}} = D_v \frac{\partial^2 V}{\partial x^2}$ is often added.
- Another possibility is using **nonlocal models** as presented next

Nonlocal second-order models

Instead of spatial derivatives, nonlocal models introduce anticipation explicitely into the acceleration function:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P(\rho)}{\partial x} = f_{\text{nonloc}}(\rho, V, \rho_a, V_a)$$

where

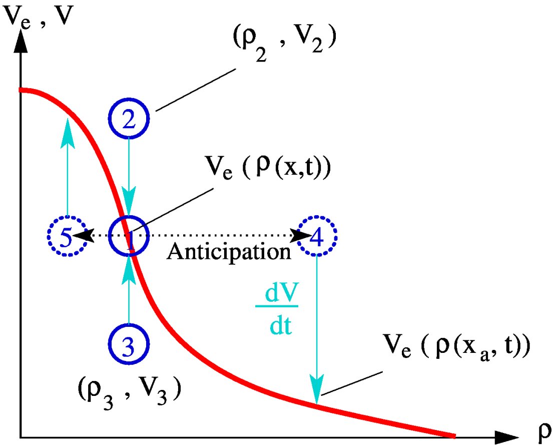
$$\rho_a = \rho(x_a, t), \quad V_a = V(x_a, t)$$

with $x_a > x$ an advanced location (model-dependent forward-looking anticipation), e.g., $x_a - x = 1/\rho$ or $= VT$

- Nonlocal models contain forward-looking explicitly, so upwind numerical differentiation (using only upstream information) is always applicable. **why?** Because downstream information is contained in the anticipated position x_a
- The "traffic pressure" $P(\rho)$ describes the same kinematic-statistical effect as in local models
- The right-hand side can be written as a nonlocal relaxation:

$$f_{\text{nonloc}}(\rho, V, \rho_a, V_a) = \frac{V_e^*(\rho, V, \rho_{\text{a}}, V_{\text{a}}) - V}{\tau}$$

Relaxation and nonlocal anticipation



- The local relaxation is the same as in local models, $f = (V(\rho) - V)/\tau$.
- The nonlocal relaxation is just $f_{\text{antic}} = (V(\rho_a) - V)/\tau$. No further approximation via Taylor series $(V(\rho_a) = V(\rho) + V'(\rho)\frac{\partial \rho}{\partial x}\,(x_a - x))$, hence no density gradients needed.

7.2 Plausibility Criteria

Introduce the (commonly used) abbreviations $V_x \equiv \frac{\partial V}{\partial x}$, $V_{xx} \equiv \frac{\partial^2 V}{\partial x^2}$, $\rho_x = \frac{\partial \rho}{\partial x}$ etc. to rewrite local and nonlocal models (pressure term integrated into f):

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \begin{cases} f_{\text{loc}}(\rho, V, \rho_x, V_x, \rho_{xx}, \ldots) & \text{local models} \\ f_{\text{nonloc}}(\rho, V, \rho_a, V_a) & \text{nonlocal models} \end{cases}$$

1. **Response to local speed:** $\frac{\partial f_{\text{loc}}}{\partial V} < 0, \quad \frac{\partial f_{\text{nonloc}}}{\partial V} < 0$ **Why?**
2. **Response to local density:** $\frac{\partial f_{\text{loc}}}{\partial \rho} \le 0, \quad \frac{\partial f_{\text{nonloc}}}{\partial \rho} \le 0$ **Why?**
3. **Homogeneous steady state:** The implicit relations

$$0 = f_{\text{loc}}\big(\rho, V_e(\rho), 0, 0, \ldots\big), \quad 0 = f_{\text{nonloc}}\big(\rho, V_e(\rho), \rho, V_e(\rho)\big)$$

leads to a steady-state speed function obeying

$$V_e(0) = V_0 = \text{max}, \quad V_e'(\rho) \le 0, \quad V_e(\rho_{\text{max}}) = 0$$

Why? The steady state is valid for all ρ. Hence $0 = \frac{\mathrm{d}f}{\mathrm{d}\rho} = \frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial V} V_e'(\rho)$, so $V_e'(\rho) = -\frac{\partial f}{\partial \rho} / (\frac{\partial f}{\partial V}) \le 0$. The maximum V_0 is reached at zero density, the value $V_e(0)$ at maximum density

7.2 Plausibility Criteria II

4. **Response to density and speed gradients:**

$$\frac{\partial f_{\text{loc}}}{\partial \rho_x} \leq 0, \quad \frac{\partial f_{\text{loc}}}{\partial V_x} \geq 0$$

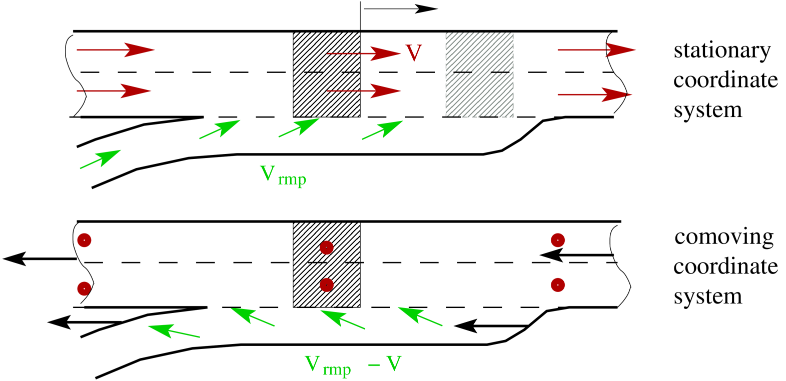
"Decelerate if the local density is increasing or the local speed is decreasing"

5. **Response to nonlocalities:**

$$\frac{\partial f_{\text{nonloc}}}{\partial \rho_a} \leq 0, \quad \frac{\partial f_{\text{nonloc}}}{\partial V_a} \geq 0$$

"Decelerate if the density ahead is larger or the speed ahead is smaller"

7.3 Ramp Terms



- The inflow/outflow of vehicles from/to ramps is modelled by the ramp term $\nu(x) = Q_{\text{rmp}}/L_{\text{rmp}}$ of the density equation. **Why?** Because the conservation of the vehicles is *always* valid
- Inflowing/outflowing vehicles that are slower than the main-road vehicles when entering/leaving cause an additional ramp term A_{rmp} in the speed equation
- To derive it, we need to consider the rate of change of the local speed in the grey box in above figure

Derivation of the on-ramp term

Rate of local speed change in the gray box of width Δx
($E(.)$ denotes the expectation value):

$$A_{\text{rmp}} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(E\left(v_\alpha\right)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n(t)}\sum_{i=1}^{n(t)} v_i\right).$$

Assuming no acceleration of the main-road and ramp vehicles (**why?**), the expectation value changes only due to vehicles entering the ramp (the off-ramp case leads to the same term if the vehicles brake on the main road to V_{rmp})

$$n = \rho L \Delta x, \quad \frac{\mathrm{d}n}{\mathrm{d}t} = Q_{\text{rmp}}\frac{\Delta x}{L_{\text{rmp}}}, \quad \sum_{i=1}^{n(t)} v_i = nV, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} v_i\right) = V_{\text{rmp}}\frac{\mathrm{d}n}{\mathrm{d}t}$$

$$\begin{aligned}
\Rightarrow A_{\text{rmp}} &= -\frac{1}{n^2}\left(\frac{\mathrm{d}n}{\mathrm{d}t}\right) nV + \frac{1}{n} V_{\text{rmp}}\frac{\mathrm{d}n}{\mathrm{d}t} \\
&= \frac{V_{\text{rmp}} - V}{n}\,\frac{\mathrm{d}n}{\mathrm{d}t} \\
&\overset{n=\rho L \Delta x}{=} \frac{V_{\text{rmp}} - V}{\rho L\, L_{\text{rmp}}}\, Q_{\text{rmp}} \\
&= \nu\left(\frac{V_{\text{rmp}} - V}{\rho}\right), \quad \nu = \frac{Q_{\text{rmp}}}{L\, L_{\text{rmp}}}
\end{aligned}$$

7.4 Specific Models I: Payne's Model

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{V_e(\rho) - V}{\tau} + \frac{V_e'(\rho)}{2\rho\tau}\frac{\partial \rho}{\partial x} \qquad \text{Payne's model}$$

- Homogeneous steady state: $V(\rho) = V_e(\rho)$ where $V_e(\rho)$ can be chosen as in the LWR model (plausibility criteria?)
- The density gradient comes from the derivation from a simple microscopic model, the **Optimal Velocity Model (OVM)** $\mathrm{d}v_i \,/\, \mathrm{d}t = (v_{\text{opt}}(s) - v)/\tau$ by the anticipation mechanism shown in 7.1: "Relaxation and nonlocal anticipation"

$$v_{\text{opt}}(s) \to V_e(\rho(x + \frac{\Delta x}{2}, t)) \approx V_e(\rho(x,t)) + V_e'\frac{\partial \rho}{\partial x}\frac{\Delta x}{2} = V_e + \frac{V_e'}{2\rho}\frac{\partial \rho}{\partial x}$$

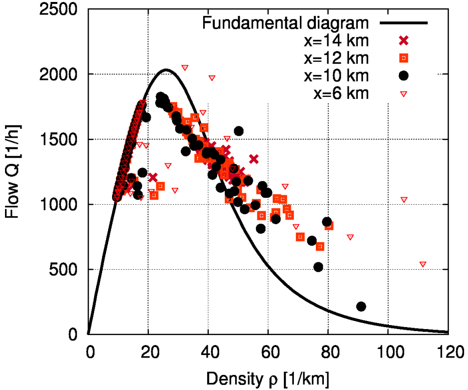
- The density gradient can also be written as a pressure term $-1/\rho\frac{\partial P}{\partial x}$ with $P = (V_0 - V_e(\rho))/(2\tau)$
- Only one parameter besides those in $V_e(\rho)$: Speed relaxation time τ of the order of 10 s

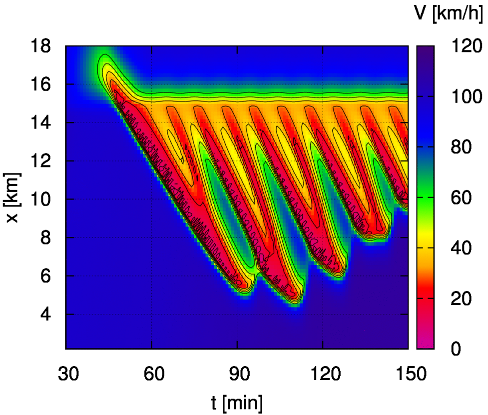
II: Kerner-Konhäuser Model

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{V_e(\rho) - V}{\tau} - \frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 V}{\partial x^2} \qquad \text{KK model}$$

- Heuristic model; no microscopic derivation; analogies to 1d compressible gas
- Same homogeneous steady state as Payne's model: $V = V_e(\rho)$
- The density gradient term is similar as in Payne's model and can be written in terms of a traffic pressure $P = c_0^2\rho$
- Additional "speed diffusion term" to avoid shock waves
- Three parameters besides that in $V_e(\rho)$ (typical values):
  - Relaxation time τ (10-30 s)
  - Sonic speed c_0 (15 m/s)
  - Speed diffusion factor μ (150 m/s)

On-ramp simulation of the KK model





- Used Speed density relation ($V_0 = 120\,\mathrm{km/h}$):

$$V_e(\rho) = V_0 \frac{1 - \rho/\rho_{\text{max}}}{1 + 200(\rho/\rho_{\text{max}})^4}$$

- The higher τ, the more prone to flow instabilities. Here, $\tau = 30\,\mathrm{s}$

III: Aw-Rascle Model

$$\frac{\partial}{\partial t}(V + p(\rho)) + V\frac{\partial}{\partial x}(V + p(\rho)) = 0 \qquad \text{Aw-Rascle model}$$

- Mathematicians love this model because it can be reformulated in totally conservative form allowing some analytic solutions:

$$\frac{\partial}{\partial t}(\rho(V + p(\rho))) + \frac{\partial}{\partial x}(\rho V(V + p(\rho))) = 0$$

- $p(\rho)$ (not the traffic pressure!) increases with speed. Often, $p(\rho) = (V_0 - V_e(\rho))$ is used (**ARZ model**)
- Its time derivative is somewhat peculiar. Using the (always valid!) continuity equation, it can be written as

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = -\rho p'(\rho)\frac{\mathrm{d}V}{\mathrm{d}x}$$

- This model does not have a FD (**why?**). Fur use in traffic flow simulation, a relaxation term $(V_e(\rho) - V)/\tau$ must be added

IV: Gas-Kinetic Based Traffic-flow (GKT) Model

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P(\rho)}{\partial x} = \frac{V_e^*(\rho, V, \rho_a, V_a) - V}{\tau} \qquad \text{GKT Model}$$

Nonlocal model with anticipated locations: $x_a = x + \gamma V T$

From the gas-kinetic derivation comes the following:

- "Traffic pressure" $P(\rho) = \rho\alpha(\rho)V_e^2$, variation coefficient $\sqrt{\alpha}(\rho)$ from the data
- Target (generally not steady-state) speed

$$V_\text{e}^*(\rho, V, \rho_\text{a}, V_\text{a}) = V_0\left[1 - \frac{\alpha(\rho)}{\alpha(\rho_\text{max})}\left(\frac{\rho_\text{a} V T}{1 - \rho_\text{a}/\rho_\text{max}}\right)^2 B\left(\frac{V - V_\text{a}}{\sqrt{2\alpha(\rho)}V}\right)\right]$$

- "Boltzmann factor" (see a statistical derivation)

$$B(x) = 2\left[x f_N(x) + (1 + x^2)\Phi(x)\right] \quad \text{(notice } B(0) = 1\text{)}$$

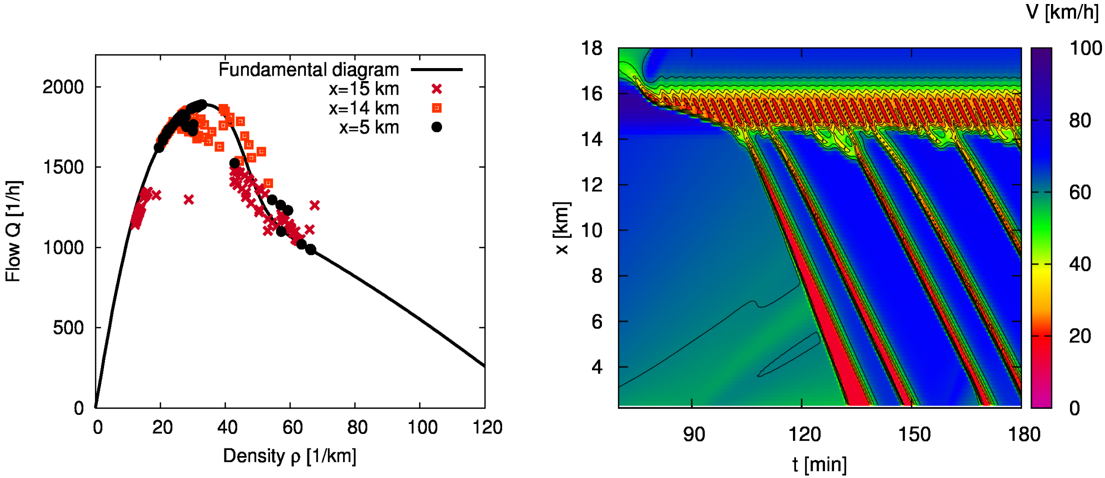
Model IV: Properties of the GKT Model

- In spite of its complexity, it is numerically stable and can be simulated efficiently
- No explicit FD, but can be calculated implicitly:

$$\begin{aligned} V_e^*(\rho, V, \rho, V) &= V, \\ V_0 - V &= \frac{\alpha(\rho)}{\alpha(\rho_{\text{max}})} \left(\frac{\rho_{\text{a}} V T}{1 - \rho_{\text{a}}/\rho_{\text{max}}} \right)^2 \\ &\Rightarrow \text{quadratic equation for } V = V_e(\rho) \end{aligned}$$

Parameter	Typical Value Highway	Typical Value City Traffic
Desired speed V_0	120 km/h	50 km/h
Desired time gap T	1.2 s	1.2 s
Maximum Density ρ_{max}	160 vehicles/km	160 vehicles/km
Speed adaptation time τ	20 s	8 s
Anticipation factor γ	1.2	1.0
variation coefficient $\sqrt{\alpha}(\rho)$	from data	(around 0.1)

Off-ramp-on-ramp simulation of the GKT model



- Off-ramp at $x = 14\,\mathrm{km}$, on-ramp at $x = 16\,\mathrm{km}$
- Solid line left image: GKT fundamental diagram
- Flow instabilities grow with increasing τ, decreasing V_0, decreasing γ and decreasing sensitivity $\alpha^{-1/2}$ (increasing speed variation coefficient)

7.5 Numerics

Essential parts of the equations of second-order models are conservative:

- Conservation of the number of vehicles in the continuum equation
- Conservation of momentum at the left-hand side of the speed equation $\Rightarrow$ take account of this in the numerical solution!

In addition, there are source terms:

- Ramps or change of the number of lanes in the continuity equation,
- Vehicle accelerations or decelerations as well as ramp source terms in the speed equation

Conservation form of the speed equation: the left-hand side

Because it is crucial for numerical accuracy to satisfy the conservation laws, reformulate the velocity equation as a flow equation: Replace in the general local or nonlocal formulation V by Q/ρ, apply the continuity equation to get rid of the appearing $\frac{\partial \rho}{\partial t}$:

$$
\begin{aligned}
\rho\ (\text{lhs.}) &= \rho \left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} \right) \\
&= \frac{\partial (\rho V)}{\partial t} - V \frac{\partial \rho}{\partial t} + \rho V \frac{\partial V}{\partial x} + \frac{\partial P}{\partial x} \\
&\overset{\text{cont.}}{=} \frac{\partial Q}{\partial t} + V \frac{\partial Q}{\partial x} + Q \frac{\partial V}{\partial x} + \frac{\partial P}{\partial x} \\
&= \frac{\partial Q}{\partial t} + \frac{\partial (QV)}{\partial x} + \frac{\partial P}{\partial x} \\
&= \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{\rho} + P \right)
\end{aligned}
$$

Conservation form: right-hand side and result

- Everything that will become a complete derivative of x in this formulation should appear on the left-hand side. This is also true for the diffusion term of the KK model becoming $-\frac{\partial}{\partial x}\left(\mu \frac{\partial Q/\rho}{\partial x}\right)$
- rhs: just redefine the remaining parts of $\rho f_{\text{loc}}(\rho, V, \ldots)$ or $\rho f_{\text{nonloc}}(\rho, V, \ldots)$ (including ramp terms) to be the source $S(\rho, Q, \rho_x, Q_x \rho_a, Q_a)$ (there should be as few gradients as possible)

Together with the continuity equation with bottlenecks, the general result is

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= \nu_{\text{rmp}} - \frac{Q}{I}\frac{\mathrm{d}I}{\mathrm{d}x} \\
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{\rho} + P - \mu \frac{\partial}{\partial x}\left(\frac{Q}{\rho}\right)\right) &= S(\rho, Q, \rho_x, Q_x, \rho_a, Q_a)
\end{aligned}
$$

With $\boldsymbol{u} = \begin{pmatrix} \rho \\ Q \end{pmatrix}$, $\quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} Q \\ \frac{Q^2}{\rho} + P - \mu \ldots \end{pmatrix}$, $\quad \boldsymbol{s}(\boldsymbol{u}) = \begin{pmatrix} \nu_{\text{rmp}} - \frac{Q}{I}\frac{\mathrm{d}I}{\mathrm{d}x} \\ S \end{pmatrix}$:

$$
\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial\ \boldsymbol{f}(\boldsymbol{u})}{\partial x} = \boldsymbol{s}(\boldsymbol{u})
$$

Upwind and McCormack Scheme

- The **upwind method** approximates $\frac{\partial f}{\partial x}$ by asymmetric first-order finite differences using upstream information (as in the LWR model for free traffic):

$$\boldsymbol{u}_k^{n+1} = \boldsymbol{u}_k^n - \frac{\Delta t}{\Delta x}(\boldsymbol{f}_k^n - \boldsymbol{f}_{k-1}^n) + \Delta t\, \boldsymbol{s}_k^n$$

  It is useful for nonlocal models since the anticipated variables ρ_a and V_a in s_k^n ensure using the upstream information

- The **McCormack method** includes two steps:
  1. calculating a **predictor** using upwind finite differences,
  2. calculating a **corrector** using downwind differences:

$$\tilde{\boldsymbol{u}}_k^{n+1} = \boldsymbol{u}_k^n - \frac{\Delta t}{\Delta x}(\boldsymbol{f}_k^n - \boldsymbol{f}_{k-1}^n) + \Delta t\, \boldsymbol{s}_k^n \qquad \text{predictor}$$

$$\boldsymbol{u}_k^{n+1} = \frac{1}{2}\left(\tilde{\boldsymbol{u}}_k^{n+1} + \boldsymbol{u}_k^n - \frac{\Delta t}{\Delta x}(\tilde{\boldsymbol{f}}_{k+1}^{n+1} - \tilde{\boldsymbol{f}}_k^{n+1}) + \Delta t\, \tilde{\boldsymbol{s}}_k^{n+1}\right) \qquad \text{corrector}$$

Approximating Nonlocalities

Assume you want to approximate $(\rho_a)_k^n$ for cell k (position $x = k\,\Delta x$) at time $t = n\,\Delta t$:

- Given the spatial anticipation horizon s_a, determine the number K of integer cells this corresponds to:

$$K = \left\lfloor \frac{s_\text{a}}{\Delta x} \right\rfloor.$$

  (typical, $K = 0$ or $=$1)

- do a piecewise linear interpolation:

$$(\rho_\text{a})_k^n \approx \rho_{k+K}^n + \left(\rho_{k+K+1}^n - \rho_{k+K}^n\right)\left(\frac{s_\text{a}}{\Delta x} - K\right)$$

- Near the downstream boundary, just use the most downstream information available

Numerical Instabilities

- Numerical instabilities have nothing to with real flow instabilities that are possible in second-order models
- Compared to the LWR numerics, there are more types of possible instabilities:
  - **convection** instabilities as in the LWR
  - **diffusive** instabilities
  - **relaxational** instabilities
  - **nonlinear** instabilities
- An analysis is only possible in the linear case $\rightarrow$ linearize the continuity and speed equations in the conservative form ($\boldsymbol{w}$: deviations in ρ and Q)

$$\frac{\partial \boldsymbol{w}}{\partial t} + \mathbf{C} \cdot \frac{\partial \boldsymbol{w}}{\partial x} = \mathbf{L} \cdot \boldsymbol{w}$$

$\mathbf{C}$: **convection matrix**; $\mathbf{L}$: **relaxation matrix**

Convection instabilities

- As in the LWR, not any signal may travel through more than one cell in one timestep.
- Two independent fields → two signal velocities given by the eigenvalues of the convection matrix

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -V^2 + \frac{\partial P}{\partial \rho} & 2V + \frac{\partial P}{\partial Q} \end{pmatrix}$$

- Calculation of the eigenvalues $c_{1/2}$ is easy if (as often) $\frac{\partial P}{\partial Q} = 0$ (and always $\frac{\partial P}{\partial \rho} \geq 0$)

$$c_{1/2} = V \pm \sqrt{\frac{\partial P}{\partial \rho}}$$

- Convection instability is avoided (for the upwind and McCormack methods) if the **first Courant-Friedrichs-Lévy (CFL) condition**

$$\Delta t < \frac{\Delta x}{\max(|c_1|, |c_2|)}$$

is satisfied for all possible V and ρ

Diffusive instabilities

Just consider the KK model, the only one with a diffusion term. In non-conservative form (no change when using the conservative form) we have with $\nu = \mu/\rho$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \ldots + \nu \frac{\partial^2 V}{\partial x^2} \approx \ldots + \nu \frac{V_{ki1}^n - 2V_k^n + V_{k-1}^n}{\Delta x^2}$$

Euler update:

$$V_k^{n+1} \approx V_k^n + \nu \Delta t \, \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\Delta x^2} + \text{other terms}$$

How would oscillating speed data $V_k^n = V_e + A(-1)^k$ be updated?
show that, in the next step, we would have

$$V_k^{n+1} = V_e + A \left(1 - \frac{4\nu\Delta t}{(\Delta x)^2} \right) (-1)^k.$$

Result: The **second CFL condition**

$$\Delta t < \frac{(\Delta x)^2}{2\nu}$$

must be satisfied for all possible V and ρ (ν may depend on ρ or V)

Relaxational instabilities for the Euler update

- For the relaxational instabilities, we need the eigenvalues of the matrix **L**. Without road inhomogeneities, we have

$$\mathbf{L} = \begin{pmatrix} 0 & 0 \\ \frac{\partial S}{\partial \rho} & \frac{\partial S}{\partial Q} \end{pmatrix}$$

which has the eigenvalues $\lambda_1 = 0$ (**plausible?**) and $\lambda_2 = \frac{\partial S}{\partial Q}$

- Obviously, $\lambda_2 > 0$ means real instability ("the faster I am with respect to the steady-state speed, the more I accelerate"). However, numerical instabilities arise for $\lambda_2 < 0$ if $1 + \Delta t \lambda_2 < -1$ (**why?**):

$\Delta t < 2 / \left\lvert \frac{\partial S}{\partial Q} \right\rvert$	**Relaxational stability criterion**
$\Delta t < 1 / \left\lvert \frac{\partial S}{\partial Q} \right\rvert$	no spurious oscillations

- For Payne's model and the KKL model, we have the source term

$$S = \frac{Q_e(\rho) - Q}{\tau} \Rightarrow \frac{\partial S}{\partial Q} = -1/\tau \;\Rightarrow\; \Delta t < \frac{2}{\tau}$$

- For the GKT model, relaxational instabilities become a problem near ρ_{max} since then $\left|\frac{\partial S}{\partial Q}\right|$ becomes large

Nonlinear instabilities

- All of the above needs linearity for its derivation
- Usually, we are nonlinear (e.g., traffic waves). You need to just look what happens ;-)
- However, the linear limits give a good guess and their negation at least is a *sufficient* criterion for instabilities!

Numerical diffusion for the Euler update

Numerical instabilities are the worst but also numerical diffusion is unwanted: To analyse, let's assume that

- the exact state $\boldsymbol{u}(x,t)$ is given at time $t = n\Delta t$ and the grid points $\boldsymbol{u}_k^n$ are exact as well,
- the flow-conservative part $\boldsymbol{f}(\boldsymbol{u})$ is at least twice differentiable in x and t,
- the convective information flow is in driving direction, so we use upswind finite differences,
- the second-order model is stripped to the bare minimum $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u}) = 0$ with $\boldsymbol{u}_t = \frac{\partial \boldsymbol{u}}{\partial t}$ (and later on $\boldsymbol{u}_x = \frac{\partial \boldsymbol{u}}{\partial x}$, $\boldsymbol{u}_{xx} = \frac{\partial^2 \boldsymbol{u}}{\partial x^2}$)

Numerical diffusion for the Euler update (ctnd)

Let's develop both the true solution and the upwind approximation for $\boldsymbol{u}(t+\Delta t, k\Delta x)$ to second order in Δx and Δt:

True solution:

$$\boldsymbol{u}(t+\Delta t, k\Delta x) \approx \boldsymbol{u} + \boldsymbol{u}_t \Delta t + \frac{1}{2}\boldsymbol{u}_{tt}(\Delta t)^2 = \boldsymbol{u} - \mathbf{C}\,\Delta t\ \boldsymbol{u}_x + \frac{1}{2}\mathbf{C}^{\,2}(\Delta t)^2\,\boldsymbol{u}_{xx}$$

where $C_{ij} = \frac{\partial f_i(t)}{\partial u_j}$ is the matrix of the partial derivatives (**Hesse matrix**) already mentioned at the convection instabilities.

Upwind approximation: Typically, the eigenvalue of $\mathbf{C}$ with the largest absolute value is positive (information direction in driving direction) $\Rightarrow$ analyze upwind finite differences (always used in the GKT model):

$$\begin{aligned}
\boldsymbol{u}_k^{n+1} &= \boldsymbol{u}_k^n - \mathbf{C}\,\frac{\boldsymbol{u}_k^n - \boldsymbol{u}_{k-1}^n}{\Delta x}\Delta t \\
&\approx \boldsymbol{u} - \frac{\mathbf{C}\,\Delta t}{\Delta x}\left(\boldsymbol{u} - \boldsymbol{u} + \boldsymbol{u}_x \Delta x - \frac{1}{2}\,\boldsymbol{u}_{xx}(\Delta x)^2\right) \\
&\approx \boldsymbol{u} - \mathbf{C}\,\Delta t \boldsymbol{u}_x + \frac{\mathbf{C}}{2}\Delta t \Delta x\,\boldsymbol{u}_{xx}
\end{aligned}$$

Numerical diffusion for the Euler update (ctnd)

The *numerical diffusion* is just the difference between the numerical and true solution in second order:

$$\frac{\boldsymbol{u}_k^{n+1} - \boldsymbol{u}(x, t+\Delta t)}{\Delta t} = \frac{1}{2}\mathbf{C}\,\Delta x \left(\mathbf{1} - \frac{\mathbf{C}\,\Delta t}{\Delta x}\right) \boldsymbol{u}_{xx} \stackrel{!}{=} D_{\text{num}}\ \boldsymbol{u}_{xx}$$

For $c_{1/2} < 0$, we need to use the downwind method leading to a sign change in the first term but the product is unchanged.

In summary, with the right upwind/downwind differentiation to avoid numerical instabilities, we have the **numerical diffusion**

$$\mathbf{D}_{\text{num}} = \frac{\Delta x}{2}\mathbf{C}\left(\mathbf{1} - \frac{\Delta t}{\Delta x}\mathbf{C}\right)$$

Remarkable: The numerical diffusion becomes very small just at the first CFL limit $\Delta t = \frac{\Delta x}{\max(|c_1|,|c_2|)}$ is reached