Lecture 06: Lighthill-Whitham-Richards (LWR) Models

- 6.1 General: Motivation and Equations
- 6.2 Wave Velocity
- 6.3 Shock-Wave Fronts
- 6.4 Triangular Fundamental Diagram (FD)
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6.1 General: Motivation and Equations

- We have three main macroscopic quantities: density ρ, flow Q, and local speed V.
- There is always the *static* **hydrodynamic relation** between these quantities arising directly by the definitions of ρ, Q, and V:

$$Q = \rho V$$

- Furthermore, vehicle conservation implies the *dynamic* **continuity equation**, e.g., for a homogeneous road:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial x} = 0$$

So, two model-independent relations between the three quantities are always there. *To make a macroscopic flow model that can be simulated, we need a third equation.*

LWR Equations

There are two basic possibilities to specify the missing third relation:

- **First-order** models or **Lighthill-Whitham-Richards (LWR)** models specify an additional static **traffic stream model/relation** between density and flow,
- **Second-order models** define a second dynamical equation for the speed

Inserting the fundamental diagram

$$Q(x,t) = Q_e(\rho(x,t))$$

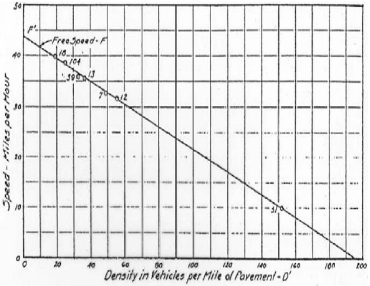
as traffic stream relation into the continuity equation gives (for homogeneous roads) the class of first-order models known as LWR models:

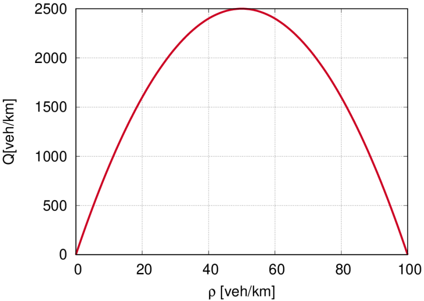
$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(Q_e(\rho)) = \frac{\partial \rho}{\partial t} + Q_e'(\rho)\frac{\partial \rho}{\partial x} = 0 \quad \text{LWR Model}}$$

? Show that the LWR model can also be written as

$$\frac{\partial \rho}{\partial t} + \left(V_e + \rho V_e'(\rho)\right)\frac{\partial \rho}{\partial x} = 0\,.$$

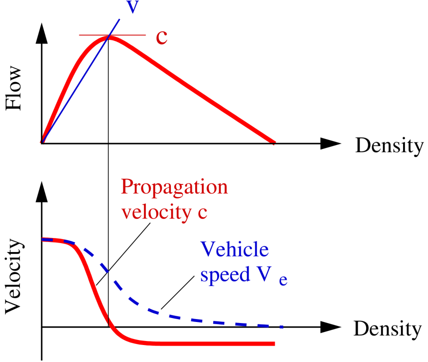
The earliest fundamental diagram of Greenshields





$$Q(\rho) = V_0 \rho \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

6.2 LWR Wave Velocity



Wave ansatz for solving the LWR equation $\frac{\partial\rho}{\partial t}+\frac{\partial Q_e(\rho)}{\partial x}=0$:

$$\rho(x,t)=\rho_0(x-ct)$$

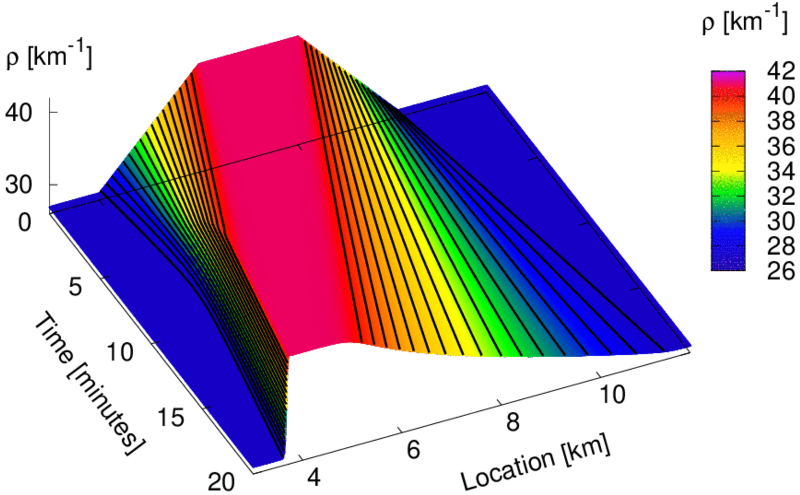
$$\frac{\partial\rho}{\partial t}=\rho_0'(x-ct)\;(-c)$$
$$\frac{\partial Q_e(\rho)}{\partial x}=Q_e'(\rho)\,\rho_0'(x-ct)$$

This solves the LWR equation for all x and t iff $\quad -c+Q_e'(\rho)=0$ or

$c=Q_e'(\rho)$ wave speed of small changes

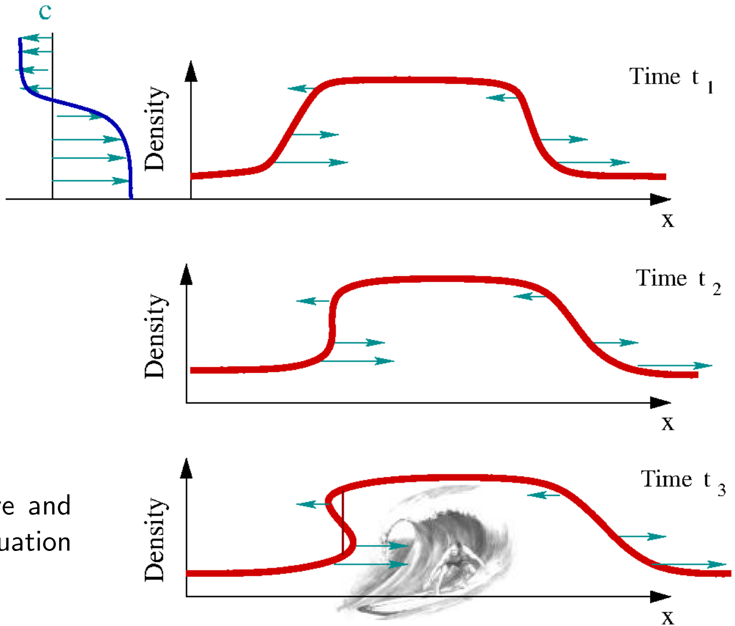
- **?** The wave speed is never larger than the vehicle speed: $c=Q_e'(\rho)=V+\rho V_e'(\rho)$. Why? base your answer on plausibility criteria
- **!** Since there are only interactions front-back but not *vice versa*

General dynamics of the LWR model



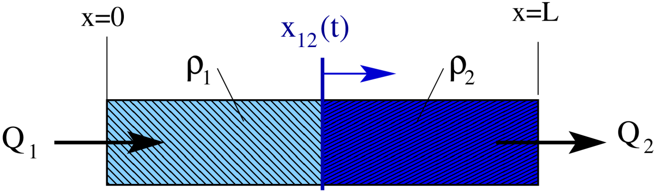
The density dependent wave speed $c = Q'_e(\rho)$ means that the density can be imagined as layers (as in a 3d printer) independently gliding over each other until a shock is formed where the solution breaks down.

6.3. Formation of Shock Waves



Both the wave and the wave equation break down!

Derivation of the shock-wave propagation velocity



- Total vehicle number: $n = \rho_1 x_{12} + \rho_2 (L - x_{12})$
- Rate of change as a function of the in- and outflows: $\frac{\mathrm{d}n}{\mathrm{d}t} = Q_1 - Q_2$
- Rate of change as partial time derivative (watch out for the moving boundary with $\frac{\mathrm{d}c_{12}}{\mathrm{d}t} = c_{12}$):

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_1 x_{12} + \rho_2 (L - x_{12})\right)\\ &= (\rho_1 - \rho_2)\frac{\mathrm{d}x_{12}}{\mathrm{d}t} = (\rho_1 - \rho_2) c_{12} \quad \Rightarrow\end{aligned}$$

$$c_{12} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = \frac{Q_e(\rho_2) - Q_e(\rho_1)}{\rho_2 - \rho_1}$$

Shock-wave equation

6.3 Problems

? Show that, in the case of a triangular fundamental diagram, the wave velocity is either equal to the vehicle speed or a constant negative value while the shock-wave propagation velocity can also takes on any value in between.

! Triangular FD: $Q(\rho) = \min(Q_f(\rho), Q_c(\rho))$;

Free traffic: $Q_f(\rho) = V_0\rho$, $c_f = Q'_f(\rho) = V_0 =$ const. (left slope);

Congested traffic: $Q_c(\rho) = 1/T(1 - \rho/\rho_{\text{max}})$, $w = Q'_c(\rho) = -1/(T\rho_{\text{max}}) =$ const. (right slope);

Shock velocity c_{12}: Slope of any line connecting the free with the congested side of the triangle, so

$c \le c_{12} \le c_f$

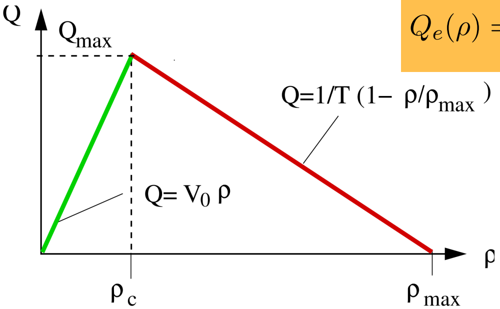
? What is the range of shock propagation velocities in the parabolic fundamental diagram of Greenshields?

! Greenshields FD: $Q(\rho) = V_0\rho(1 - \rho/\rho_{\text{max}})$, $Q'(\rho) = V_0(1 - 2\rho/\rho_{\text{max}})$

$\Rightarrow$ both the wave and the shock velocities can take on values between $Q'(\rho_{\text{max}}) = -V_0$ and $Q'(0) = +V_0$

6.4 Triangular FD: Definition and Basic Properties

$$Q_e(\rho) = \begin{cases} V_0\rho & \text{if } \rho \le \rho_\mathsf{c} \\ \frac{1}{T}(1-\rho l_\mathsf{eff}) & \text{if } \rho_\mathsf{c} < \rho \le \rho_\mathsf{max} \end{cases}$$



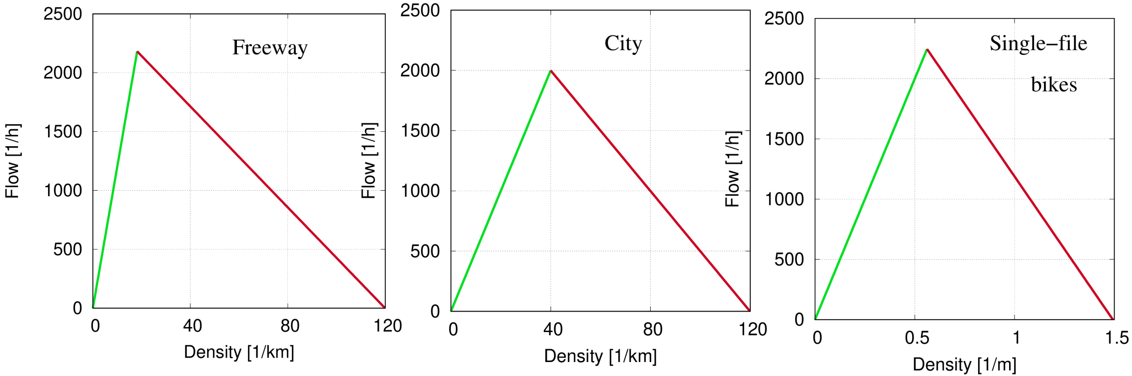
- Critical density: $\rho_c = \frac{1}{V_0T+l_\mathsf{eff}}$
- Maximum flow: $Q_\mathsf{max} = \frac{V_0}{V_0T+l_\mathsf{eff}}$
- Maximum density: $\rho_\mathsf{max} = \frac{1}{l_\mathsf{eff}}$

Model parameters:

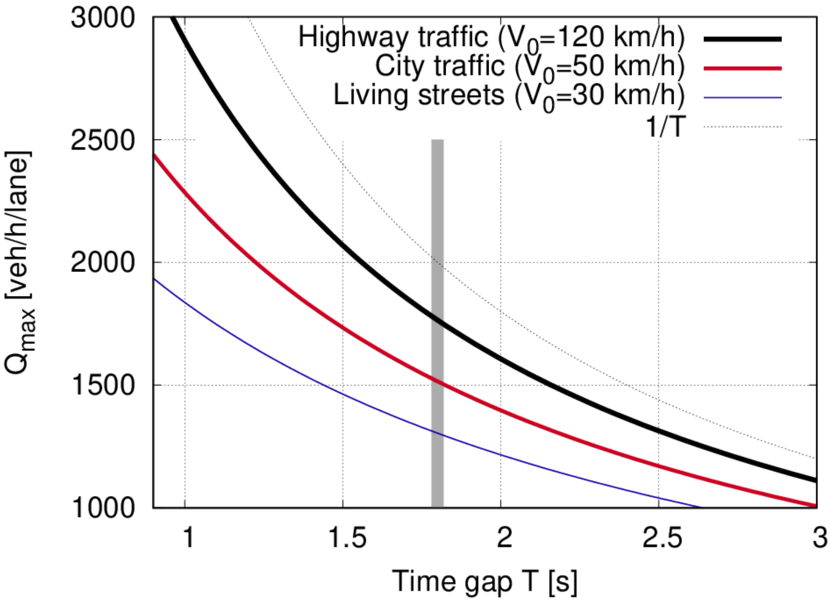
- Desired speed V_0
- Effective vehicle length l_eff or maximum density $\rho_\mathsf{max} = \frac{1}{l_\mathsf{eff}}$
- Effective time gap T or wave speed $w = -\frac{l_\mathsf{eff}}{T}$

Typical parameters in different situations

Parameter	**Highway**	**City Traffic**	**Pedestrian Single File**
Desired speed V_0	120 km/h	50 km/h	1.2 m/s
Time gap T	1.4 s	1.2 s	1.0 s
Max. density ρ_{max}	120 veh/km	120 veh/km	1.5 peds/m



Capacity as a function of the model parameters



6.5 Properties of the Triangular FD

$$Q(\rho) = \min\left(V_0\rho, \frac{1-\rho l_{\text{eff}}}{T}\right)$$

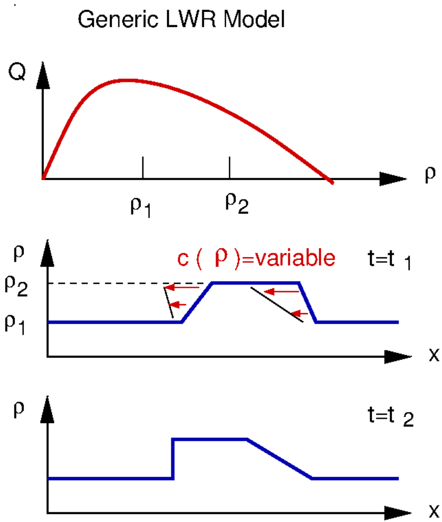
- Analytic expression for the density **at capacity**: $\rho_{\text{c}} = \frac{1}{V_0 T + l_{\text{eff}}}$
- Analytic expression for the capacity: $Q_{\text{max}} = V_0 \rho_c = \frac{V_0}{V_0 T + l_{\text{eff}}}$
- Fixed wave propagation velocities: $c(Q) = \begin{cases} V_0 & \text{free} \\ w = -\frac{l_{\text{eff}}}{T} & \text{congested} \end{cases}$
- Analytic inverse relations:

$$\rho_{\text{free}}(Q) = \frac{Q}{V_0}, \quad \rho_{\text{cong}}(Q) = \frac{1-QT}{l_{\text{eff}}} = \rho_{\text{max}}(1-QT)$$

- By means of the relations $Q_{\text{max}} = V_0/(V_0 T + l_{\text{eff}})$ and $w = -l_{\text{eff}}/T$, the unobservable quantities l_{eff} and T can be eliminated and the FD reformulated in terms of the observable parameters V_0, Q_{max} and w:

$$Q_e(\rho) = \begin{cases} V_0\rho & \text{if } \rho \le \rho_c = \frac{Q_{\text{max}}}{V_0} \\ Q_{\text{max}}\left[1 - \frac{w}{V_0}\right] + w\rho & \text{if } \rho > \rho_c \end{cases}$$

Unique Properties of the Triangular FD (2)



In the triangular FD, waves in one regime (free or congested) remain unchanged

Unique Properties of the Triangular FD (3)

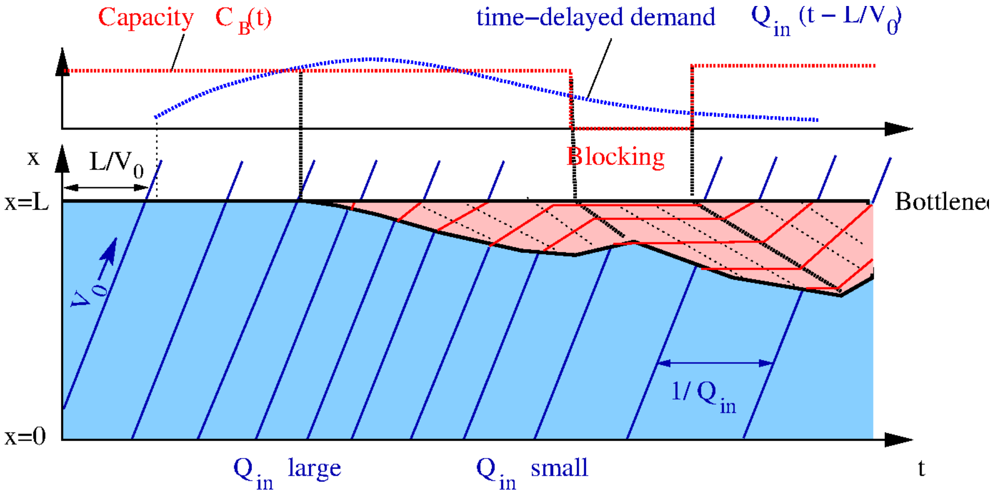
The **upstream jam front** free → congested can be calculated by a time delayed ODE without solving the whole PDE using boundary conditions (e.g., from a detector) at both ends:

$$c_{12} = \frac{\mathrm{d}x_{12}}{\mathrm{d}t} = \frac{Q_1(t-\tau_f) - Q_2(t-\tau_c)}{\rho_1(t-\tau_f) - \rho_2(t-\tau_c)}$$

- $Q_1(t)$: free traffic inflow from an upstream stationary detector
- $Q_2(t)$: congested outflow from a downstream stationary detector
- $\tau_f = (x_{12} - x_1)/V_0 > 0$:
  signal travel time from the upstream boundary to the front
- $\tau_c = (x_{12} - x_{\text{down}})/w > 0$:
  signal travel time from the downstream boundary to the front

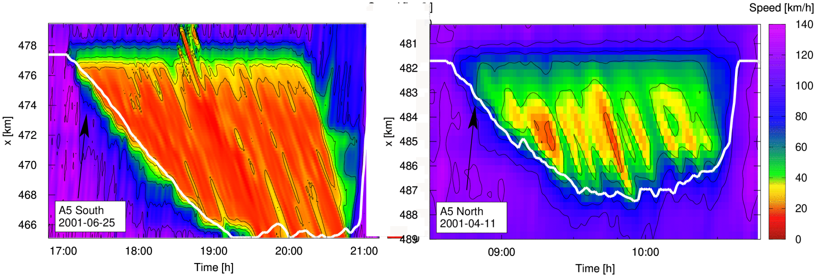
The **downstream jam front** is either fixed at a bottleneck or moves upstream at velocity w

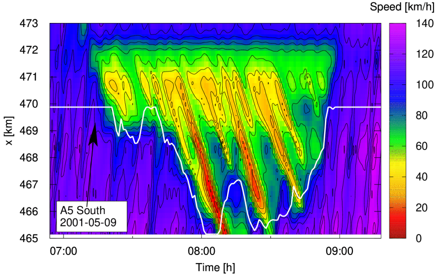
Application: State Detection and Short-Term Forecast: Upstream Jam Front



Both times τ_f and τ_c are positive ⇒ real prediction based on vehicle conservation!

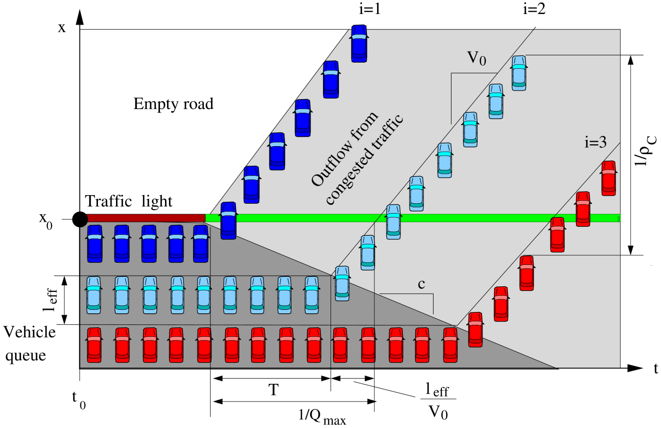
Application to three real situations (SDD available)





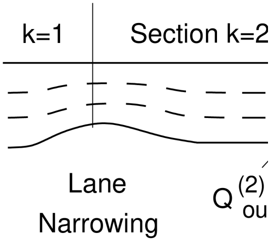
By calibrating the LWR parameters (essentially w and Q_{max} since V_0 has little influence), one obtains an *unbiased* estimate for ρ_{max}

Downstream shock front seen microscopically



- Infinite aceleration (softens for nontriangular FD)
- The upstream front is always sharp irrespective of the FD and corresponds to an infinite deceleration

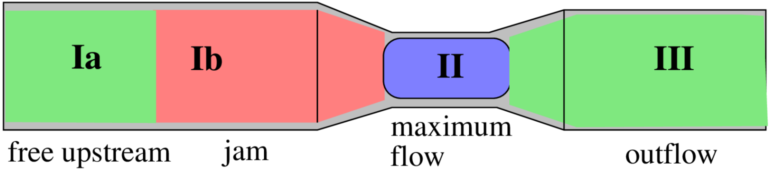
6.6 Bottlenecks – an overview



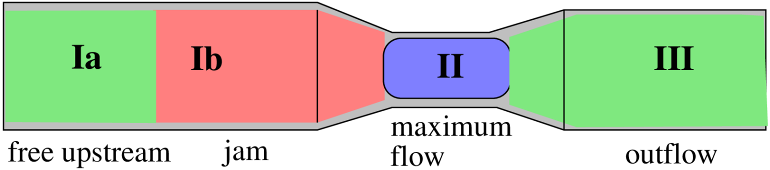
Types of bottlenecks:

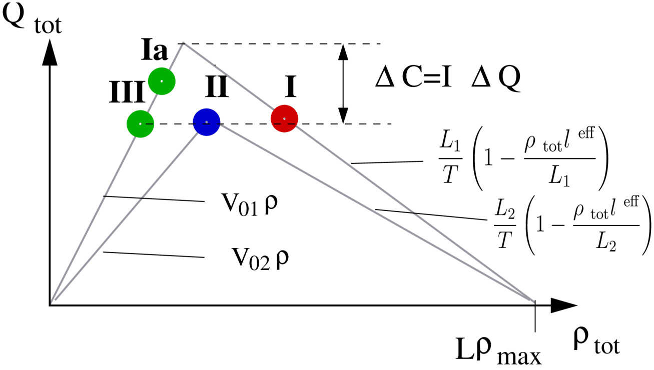
- Lane and **flow-conservative bottlenecks**: no source terms, bottleneck effect only by spatial change of parameters
- **Lane closure bottlenecks**: bottleneck effect by source terms in the effective LWR while the LWR for the total quantities has no sources (flow-conservative)
- **Ramp bottlenecks**: bottleneck effect by source terms
- **Temporary bottlenecks** such as traffic lights

Classic flow-conserving bottleneck



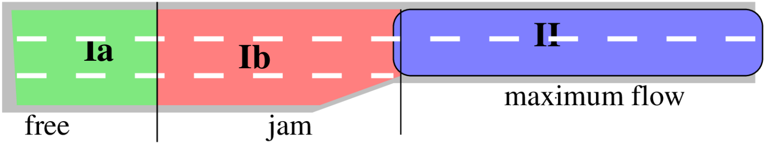
Classic flow-conserving bottleneck



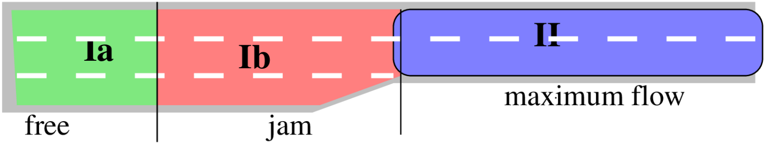


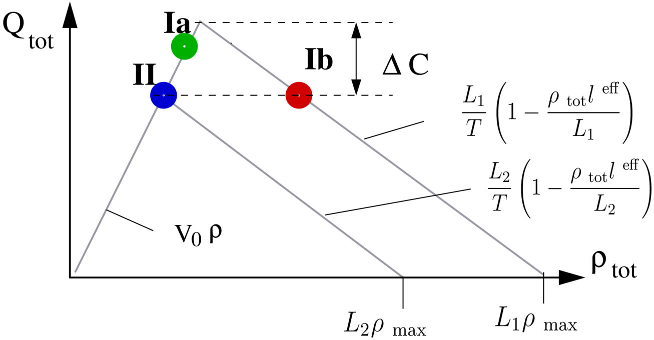
- The bottleneck with capacity $C_{\text{bottl}} = LQ_{\text{max}}$ has a different FD with a lower capacity than the other sections
- Definition of bottleneck: locally reduced capacity
- If congestion arises, the bottleneck emits information in both directions (**why?**). $\Rightarrow Q_{\text{Ib}}$ and Q_{III} are equal to the bottleneck capacity Q_{II}

Lane-closing bottleneck



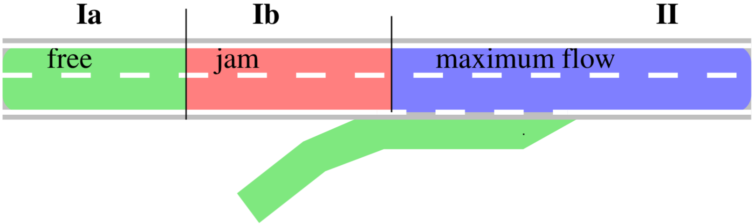
Lane-closing bottleneck



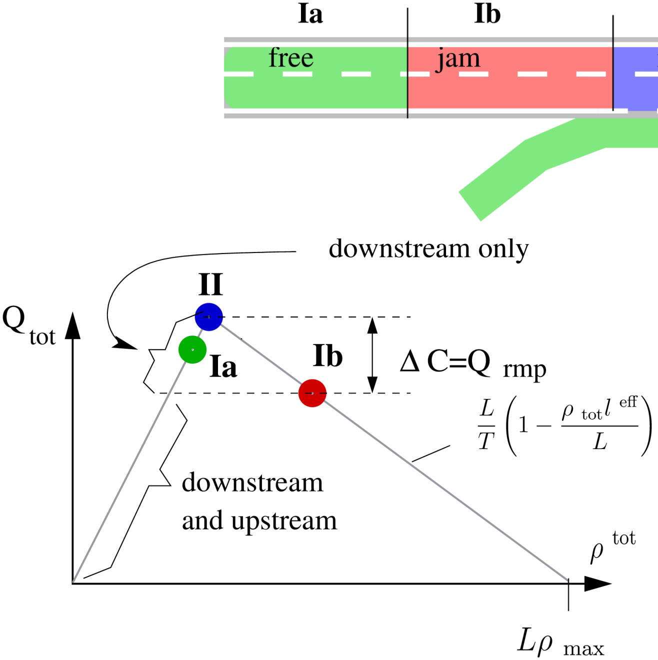


- Same FD per lane but different FD for total quantities
- Bottleneck capacity: $C_{\text{bottl}} = L_2 Q_{\text{max}}$
- Maximum-flow state in the whole downstream region, in contrast to the classic bottleneck

On-ramp bottleneck

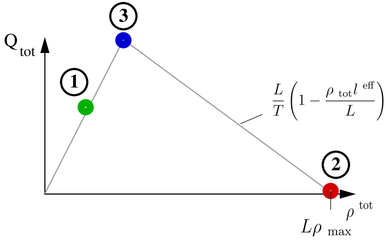


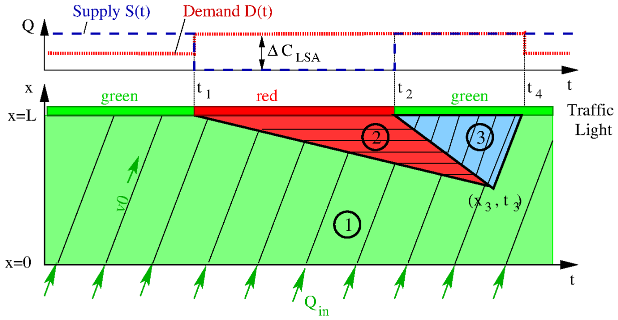
On-ramp bottleneck



- All road sections have the same FD but there is a source term in the continuity equation
- Bottleneck capacity $LQ_{\text{max}} - Q_{\text{rmp}}$
- Continuous transition congested-maximum flow state in merging region

6.7 Traffic Lights: Temporary bottlenecks

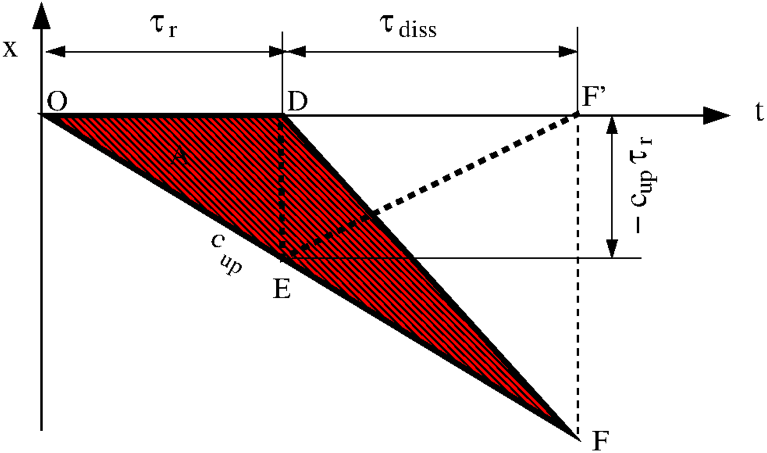




- In the LWR approach, a traffic light is a temporry bottleneck with capacity zero
- When the light becomes green, the stationary downstream jam front congested → free starts to move upstream at velocity w
- Generally, downstream fronts are either "pinned" at the bottleneck or move upstream at velocity w

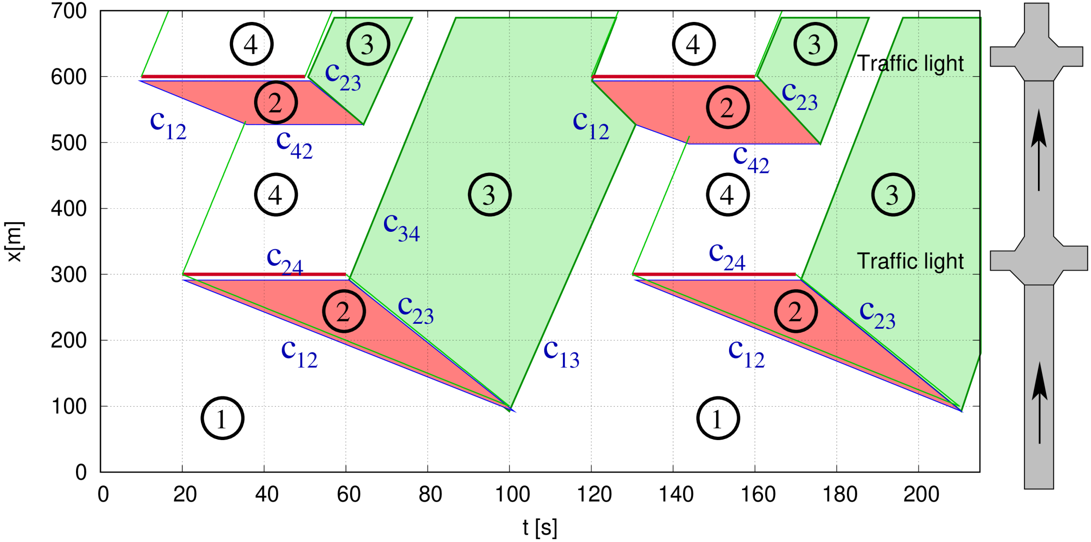
? There is a single situation where a downstream front may move downstream. Which?
Moving bottleneck, e.g., a heavy transport

Calculation of the total waiting time in one red phase



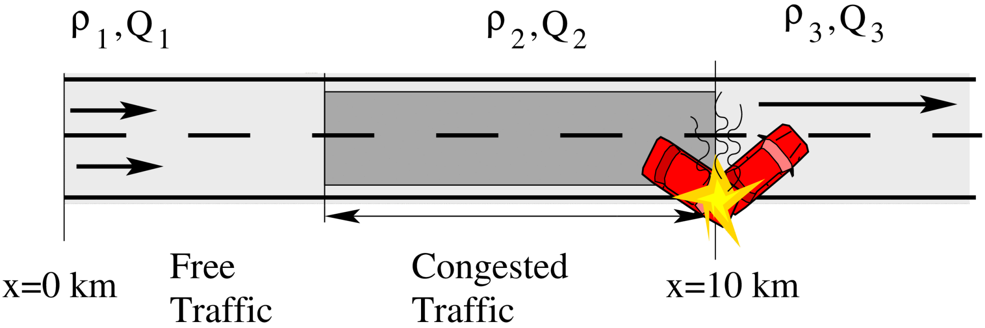
- ρ_{max} times the vertical extension of the red triangle (e.g., from E to D) gives the actual number of stopped vehicles per lane
- The area of the triangle gives the total waiting time per lane
- This area can be easily calculated by adding the areas of the two rectangular triangles ODE and DEF': $T_{\text{wait,lane}} = \frac{1}{2}\rho_{\text{max}}|c_{\text{up}}|\left(\tau_{\text{r}} + \tau_{\text{diss}}\right)$

Several traffic lights



Draw a triangular FD with the traffic states ① to ④ and also denote graphically the different shock-wave propagation velocities

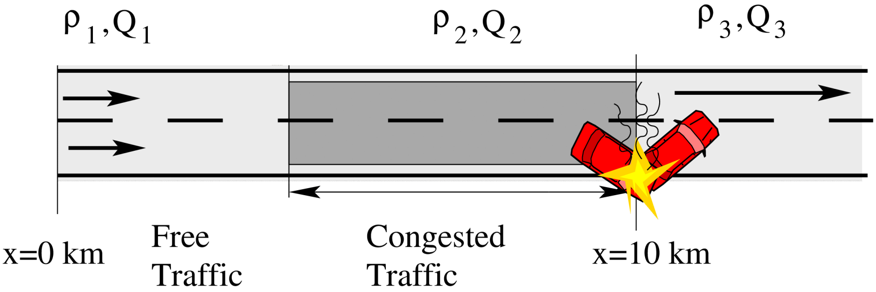
6.8 Examples: 1.accident



- Single-lane obstruction between 15:00 and 15:30 pm
- Constant inflow 3,024 veh/h
- Triangular FD parameters $l_{\text{eff}} = 8\,\text{m}$, $T = 1.5\,\text{s}$, and $V_0 = 28\,\text{m/s}$

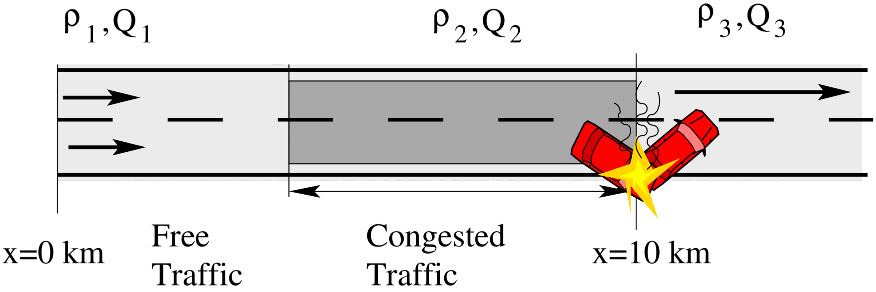
1. Does the road capacity prior to the accident satisfy the demand?

! Maximum flow per lane: $Q_{\text{max}} = \frac{V_0}{V_0 T + l_{\text{eff}}} = 2{,}016\,\text{veh/h}$.
Capacity before the accident: $C = 2 Q_{\text{max}} = 4{,}032\,\text{veh/h}$.
This exceeds the traffic demand 3,024 veh/h, so no jam.



2. Show that the accident leads to a traffic breakdown. Calculate the total and effective flows in all sections.

- Bottleneck capacity $C_{\text{bottl}} = Q_{\text{max}} = 2,016\,\text{veh/h}$ is smaller than inflow Q_{in} $\Rightarrow$ traffic breakdown.
- Upstream free flow controlled by inflow, $Q_1^{\text{tot}} = 3,024\,\text{veh/h}$, and congested flow as well as the flow in all following segments by the bottleneck: $Q_2^{\text{tot}} = Q_3^{\text{tot}} = C_{\text{bottl}} = 2,016\,\text{veh/h}$
- Per lane, the effective flows are $Q_1 = 1,512\,\text{veh/h}$, $Q_2 = Q_3 = 1,008\,\text{veh/h}$



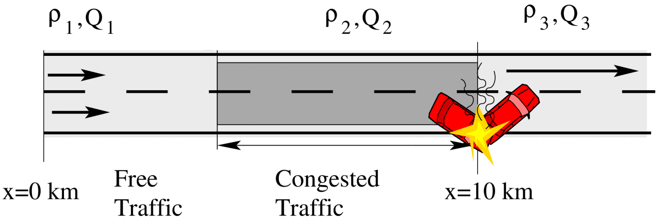
3. Calculate the propagation velocity of the upstream jam front

- Bedides the flow, we need the densities. They are given by the suitable branch of the inverse FD:

$$\rho_1 = \frac{Q_1}{V_0} = 15\,\mathrm{veh/km}, \quad \rho_2 = \rho_{\text{cong}}(Q_2) = \frac{1 - Q_2 T}{l_{\text{eff}}} = 72.5\,\mathrm{ve/km}$$

- Propagation velocity of upstream jam front:

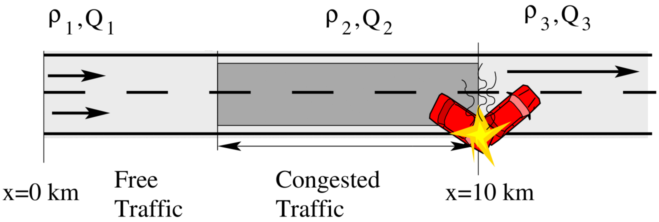
$$c^{\text{up}} = c_{12} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = -8.77\,\mathrm{km/h}$$



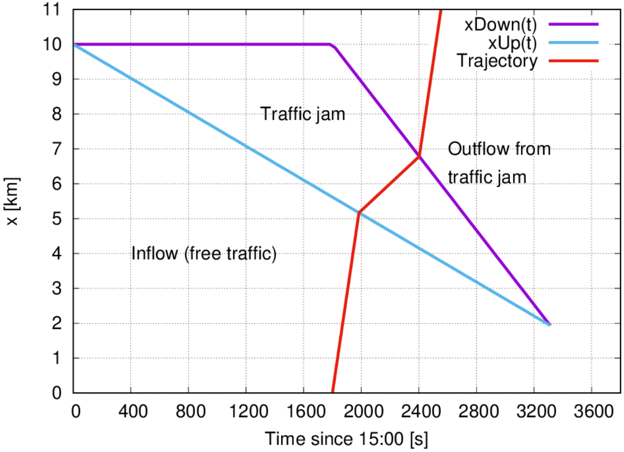
4. Calculate the velocity of the moving downstream front once the obstruction has been removed and the time for complete dissolution of the jam.

Once the obstruction has been removed, the maximum flow state always arising at the dowwnstream end of jams is over both lanes, so the new flows downstream of the congestion are $Q_4^{\text{tot}} = C$ and $Q_4 = C/2 = Q_{\text{max}} = 2,016\,\mathrm{veh/h}$ and, from the free branch, $\rho_4 = Q_4/V_0 = 20\,\mathrm{vehicles/h}$. Thus,

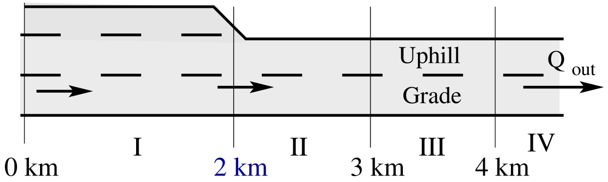
$$c^{\text{down}} = c = c_{23} = \frac{Q_4 - Q_2}{\rho_4 - \rho_2} = -19.2\,\mathrm{km/h}$$



5. Visualize the spatiotemporal dynamics of the jam by drawing its boundaries in a space-time diagram.
6. How much time does a vehicle need to traverse the 10 km long road section if it enters at 15:30 h?



Example 2: Uphill Grade and Lane Drop



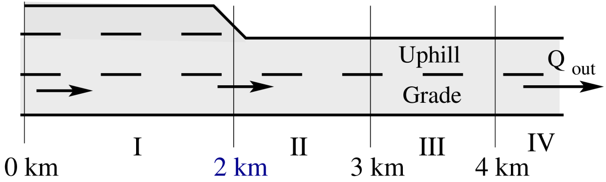
Triangular FD (per lane) as follows:

$$V_0 = \begin{cases} 60\,\mathrm{km/h} & \text{Section III} \\ 120\,\mathrm{km/h} & \text{other sections} \end{cases}, \quad T = \begin{cases} 1.9\,\mathrm{s} & \text{III} \\ 1.5\,\mathrm{s} & \text{others} \end{cases}, \quad l_{\text{eff}} = 10\,\mathrm{m}$$

1. Calculate the local road capacity and identify possiblebottlenecks

! capacity is number of lanes times Q_{max} where Q_{max} is equal for the Sections I, II, and IV; bottlenecks are local capacity drops, i.e., beginning of the Sections II and III → tutorial to this lecture

Example 2: Uphill Grade and Lane Drop



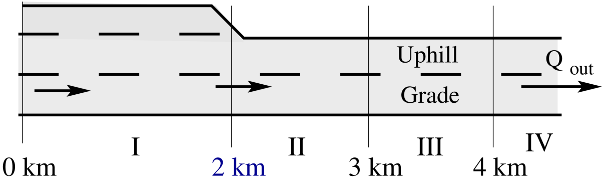
Triangular FD (per lane) as follows:

$$V_0 = \begin{cases} 60\,\text{km/h} & \text{Section III} \\ 120\,\text{km/h} & \text{other sections} \end{cases}, \quad T = \begin{cases} 1.9\,\text{s} & \text{III} \\ 1.5\,\text{s} & \text{others} \end{cases}, \quad l_\text{eff} = 10\,\text{m}$$

2. At 4:00 pm, the total traffic demand at $x = 0$ increases abruptly from 2,000 veh/h to 3,600 veh/h. Does this cause a breakdown? If so, at which time and where?

! Answer: Check where, when going from upstream to downstream (why?) the demand exceeds the capacity for the first time. The relevant bottleneck is said to be **activated** → tutorial to this lecture

Example 2: Uphill Grade and Lane Drop



Triangular FD (per lane) as follows:

$$V_0 = \begin{cases} 60\,\mathrm{km/h} & \text{Section III} \\ 120\,\mathrm{km/h} & \text{other sections} \end{cases}, \quad T = \begin{cases} 1.9\,\mathrm{s} & \text{III} \\ 1.5\,\mathrm{s} & \text{others} \end{cases}, \quad l_{\text{eff}} = 10\,\mathrm{m}$$

3. Calculate the dynamics of the developing congestion if the inflow remains constant

! The downstream front is pinned at the activated bottleneck. The upstream front is determined by the shockwave formula. You need to distinguish the propagation in the Regions II and I! → tutorial to this lecture

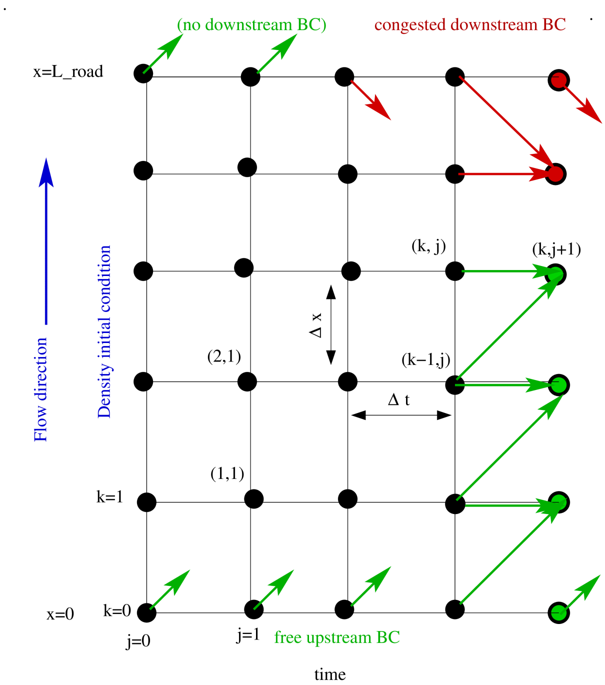
6.9 Numerics of the LWR

I: Mathematical background

Mathematically, the LWR model equation $\frac{\partial \rho}{\partial t} + \frac{\partial Q_e(\rho)}{\partial x}$ is a **hyperbolic** partial differential equation (PDE) for $\rho(x, t)$ in the special form of a **conservation law**. This PDE can be solved (the problem is *well posed*; the solution exists and is unique as the math people say) provided

- The **initial condition** $\rho(x, 0)$ at time $t = 0$ is completely known for all $x \in [0, L_{\text{road}}]$ along the road
- In case of free traffic $\rho(0, t) < \rho_c$, the **upstream boundary condition (BC)** $\rho(0, t) = \rho_{\text{free}}(Q_{\text{up}}(t))$ is given by the traffic demand Q_{up} per lane
- In case of a downstream congestion, the **downstream BC** $\rho(L_{\text{road}}, t) = \rho_{\text{cong}}(Q_{\text{down}}(t))$ is determined by the maximum flow this congestion can take.

- When solving a conservation law, it is crucial to take into account the direction of information flow.,
- Depending on the situation, 0, 1, or 2 BC apply



General LWR models

- Cell length Δx, update time interval Δt, approximated density $\rho_{kj} = \rho(k\Delta x, j\Delta t)$
- take into account signal propagation directions

General LWR models (ctnd): discretisation in space

- Approximate the spatial derivative $\frac{\partial Q}{\partial x}$ by first-order finite differences taking account the signal propagation (essentially **Godunov's method**)
- If information propagates downstream (free traffic), use **downwind finite differences** ("with the wind, with the stream") for $\frac{\partial}{\partial x}$ to "catch" this information:

$$\frac{\partial Q(k\Delta x, j\Delta t)}{\partial x} \approx \frac{Q_{k,j} - Q_{k-1,j}}{\Delta x}$$

- If information propagates upstream (congested traffic), use **upwind finite differences** ("against the wind"):

$$\frac{\partial Q(k\Delta x, j\Delta t)}{\partial x} \approx \frac{Q_{k+1,j} - Q_{k,j}}{\Delta x}$$

General LWR models (ctnd): discretisation in time

- **Explicit** numerical scheme: only past and present needed
- **First-order** scheme: errors for integrating a certain time interval decrease linearly with Δt and Δx if both tend to zero
- The **Euler method** is the simplest of such schemes: $f(t+\Delta t) \approx f(t) + f'(t)\Delta t$ for any $f(t)$

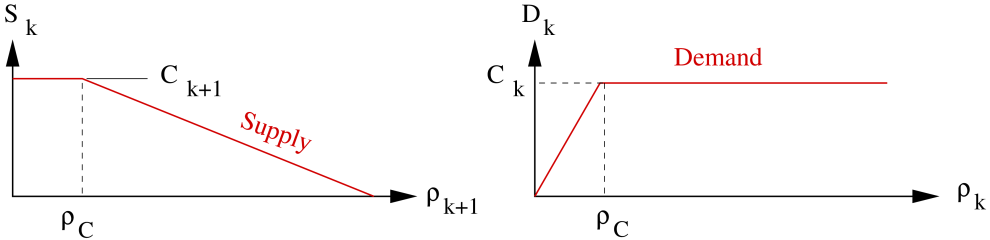
$\Rightarrow$

$$\boxed{\begin{array}{rcl} \rho_{k,j+1} & = & \rho_{k,j} - \frac{\Delta t}{\Delta x}\begin{cases} \left(Q_{k-1,j} - Q_{k,j}\right) & \text{free} \\ \left(Q_{k+1,j} - Q_{k,j}\right) & \text{congested} \end{cases}, \\ Q_{k,j+1} & = & Q_e(\rho_{k,j+1}) \end{array}}$$

Courant-Friedrichs-Lévy (CFL) stability criterion:

$$\boxed{\Delta t \le |c|_{\text{max}} \Delta x = V_0 \Delta x}$$

Numerics III: Supply-Demand-Method for Triangular FDs

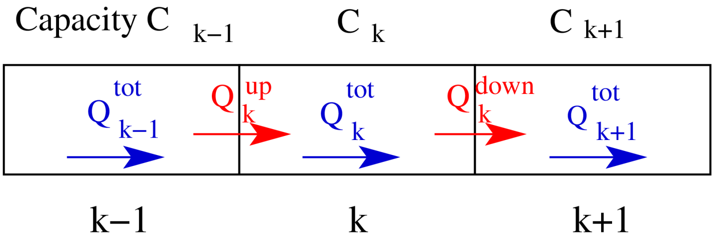


General scheme also applies to triangular FDs. However, near capacity, it becomes tricky to determine whether downwind or upwind finite differences to apply (why?). The **supply-demand method** gives a specialized simplified procedure for triangular FDs:

1. Define the supply (maximum flow the downstream cell can receive) and demand (maximum flow the upstream cell can deliver) functions with capacity $C_k = L_k Q_{\text{max}}$:

$$\begin{aligned} S(k) &= \min(C_k, L_k Q_{\text{cong}}(\rho_k)), \\ D(k) &= \min(C_k, L_k Q_{\text{free}}(\rho_k)) \end{aligned}$$

Supply-Demand-Method for Triangular FDs (ctnd)



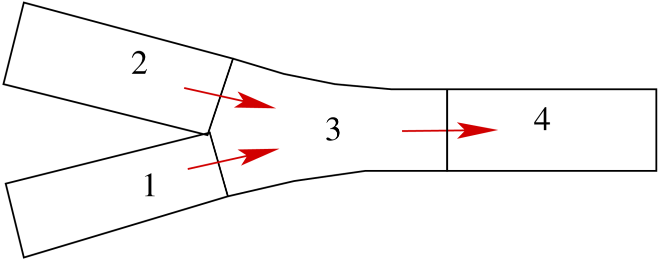
2. As in any trading, the actual flow (amount of delivered products) is given by the minimum of supply and demand. For the two boundaries of cell k:

$$\begin{aligned} Q_k^{\text{tot,up}} &= Q_{k-1}^{\text{tot,down}} = \min\left(S_k, D_{k-1}\right), \\ Q_k^{\text{tot,down}} &= Q_{k+1}^{\text{tot,up}} = \min\left(S_{k+1}, D_k\right). \end{aligned}$$

3. Explicit first-order as in the general case:

$$\begin{aligned} \rho_k(t+\Delta t) &= \rho_k(t) + \frac{1}{L_k \Delta x_k}\left(Q_k^{\text{tot,up}} - Q_k^{\text{tot,down}}\right)\Delta t, \\ Q_k(t+\Delta t) &= Q_e\left(\rho_k(t+\Delta t)\right). \end{aligned}$$

Cell-transmission model for networks



$$\begin{aligned}
S_3 &= \min(C_3, L_3 Q_{\text{cong}}(\rho_3)), \\
D_3 &= \min(C_3, L_3 Q_{\text{free}}(\rho_3)), \\
D_{12} &= \min(C_1, L_1 Q_{\text{free}}(\rho_1) + \min(C_2, L_2 Q_{\text{free}}(\rho_2), \\
Q_3^{\text{tot,up}} &= \min(S_3, D_{12}), \\
Q_3^{\text{tot,down}} &= \min(S_4, D_3), \\
\rho_3^{\text{tot}}(t+\Delta t) &= \rho_3^{\text{tot}} + \frac{1}{\Delta x_3}\left(Q_3^{\text{tot,up}} - Q_3^{\text{tot,down}}\right)\Delta t
\end{aligned}$$

In case of congestion, only the *sum* $Q_3^{\text{tot,up}}$ is defined and the supply distributed to the demands D_1 and D_2 according to traffic regulations/priority rules.

Cell transmission model for other bottlenecks

- *Bottlenecks in the stricter sense* and also changes in the *number of lanes*? are automatically included in the supply-demand model for straight roads
- *Merge* bottlenecks? are just two-in-one nodes where the priority (in contrast to regulations) is given to the ramp
- *Diverges*? such as cell 1 → cells 2, 3 require the diverging fraction λ_3 as additional input. Congestion arises if $\lambda_3 D_1 > S_3$ or $(1-\lambda_3)D_1 > S_2$. If $\lambda_3 D_1 > S_3$, we have

$$Q_1^{\text{tot,down}} = \frac{S_3}{\lambda_3}$$

leading to a **spill-back bottleneck**

1. Make this formula plausible!

On average, a flow S_3/λ_3 can pass the diverge such that Link 3 is at its supply limit S_3. The excess link-3 drivers wait until they can enter thereby obstructing also the upstream link-2 drivers

2. Discuss lane changes as sources of off-ramp bottlenecks Lane changes disturb the flow reducing the maximum flow. This cannot be modelled by LWR models unless special provisions are taken.