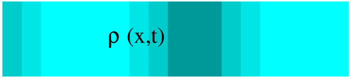
Part II: Traffic Flow Models

Lecture 05: Macroscopic Traffic Flow Models: General

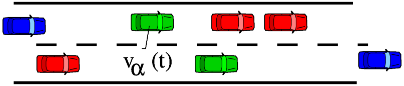
- 5.1. Model Overview
- 5.2. Macroscopic Quantities for lane-based traffic
- 5.3. Macroscopic Quantities for directed 2d traffic
- 5.4. Traffic Stream Relations
- 5.5. Hydrodynamic Relation
- 5.6. Continuity Equation
- 5.7. Eulerian *vs.* Lagrangian view

5.1. Model Overview

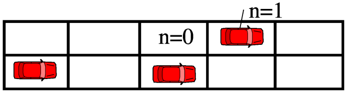
Macroscopic Model



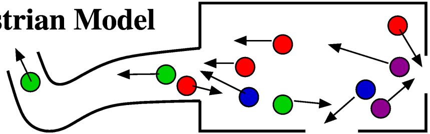
Microscopic Model



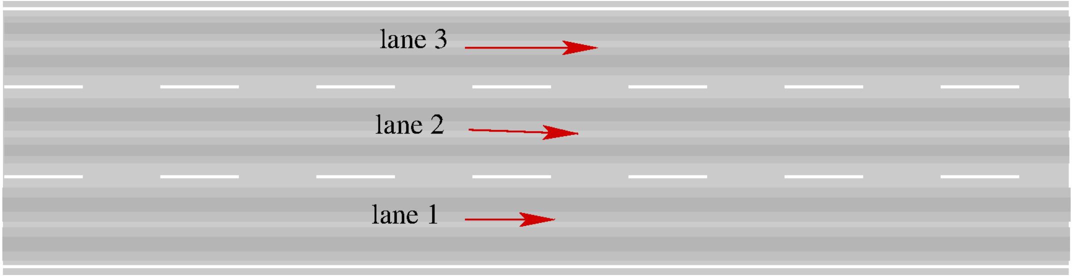
Cellular Automaton (CA)



Pedestrian Model



5.2. Basic Macroscopic Quantities for Lane-Based Traffic



Three categories of macroscopic quantities:

- per lane: ρ_l, Q_l, V_l
- total: ρ^{tot}, Q^{tot}
- effective/average: ρ, Q, V

Extensive and intensive quantities

- **Extensive** quantities (increasing with vehicle number, here ρ and Q) will just added/averaged normally to obtain total and effective values, respectively:

$$\rho^{\text{tot}} = \sum_{l=1}^{L} \rho_l, \quad \rho = \frac{1}{L}\sum_{l=1}^{L} \rho_l = \frac{\rho^{\text{tot}}}{L}, \quad Q \text{ likewise}$$

- **Intensive** quantities such as macroscopic speed V or speed variance cannot be added sensibly $\Rightarrow$ no "total" quantity. The lane averaging may also be more tricky.

? Determine V in two ways such that the macroscopic hydrodynamic relation $Q = \rho V$ and $Q^{\text{tot}} = V\rho^{\text{tot}}$ holds. Identify the results with weighted arithmetic and harmonic averages

! First, because the average and total extensive quantities only differ by the lane number L, we have $Q/\rho = Q^{\text{tot}}/\rho^{\text{tot}}$. We calculate just the ratio of the total quantities

- $V = \frac{Q^{\text{tot}}}{\rho^{\text{tot}}} = \frac{\sum_l \rho_l V_l}{\rho^{\text{tot}}} = \sum_l w_{l\rho} V_l$, $\Rightarrow$ **arithmetic** average with weighting $w_{l\rho} = \frac{\rho_l}{\rho^{\text{tot}}}$
- $V^{-1} = \frac{\rho^{\text{tot}}}{Q^{\text{tot}}} = \frac{\sum_l \rho_l}{Q^{\text{tot}}} = \frac{\sum_l \frac{Q_l}{V_l}}{Q^{\text{tot}}} = \sum_l w_{lQ} \frac{1}{V_l}$, $\Rightarrow$ **harmonic** average with weighting $w_{lQ} = \frac{Q_l}{Q^{\text{tot}}}$

5.3. Basic Directed 2d Traffic

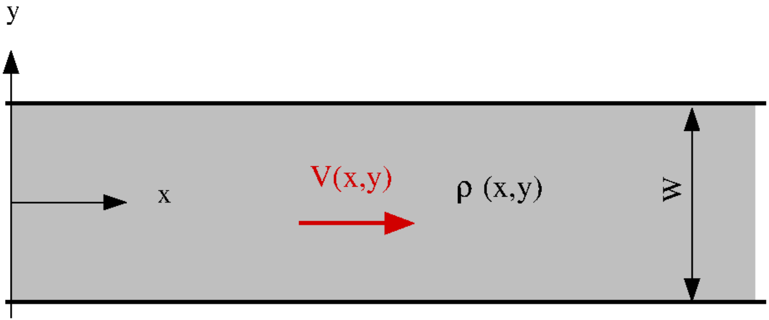
Example II: Hajj in Mekka

Traffic signs at the Hajj

Example III: Loveparade

Example IV: Vasaloppet

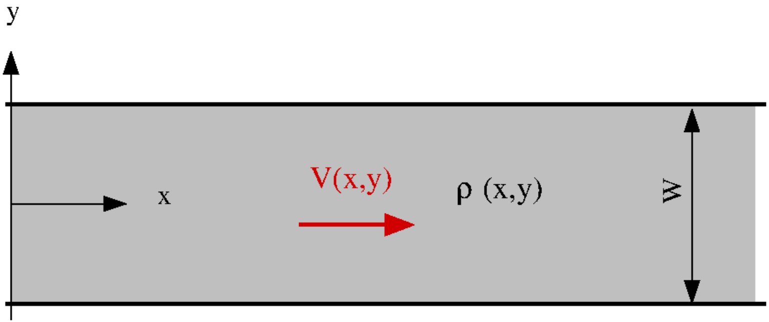
Basic macroscopic 2d quantities



- **Density** $\rho(x,y,t) = \rho(\boldsymbol{x},t)$ pedestrians per square meter $[\mathrm{ped}/\mathrm{m}^2]$
- **Flow density** $\boldsymbol{J}(\boldsymbol{x},t)$, $J(\boldsymbol{x},t) = |\boldsymbol{J}(\boldsymbol{x},t)|$ pedestrian flow per meter cross-section [ped/(ms)] ,
- **Local velocity and speed** $\boldsymbol{V}(\boldsymbol{x},t) = \boldsymbol{J}/\rho$, $V(\boldsymbol{x},t) = J/\rho$ [m/s].

Essentially, the flow density is the limit of the flow per lane divided by the lane width for a multi-lane road with the lane number going to infinity at constant width W: $\sum_l \to \int \, \mathrm{d}y$

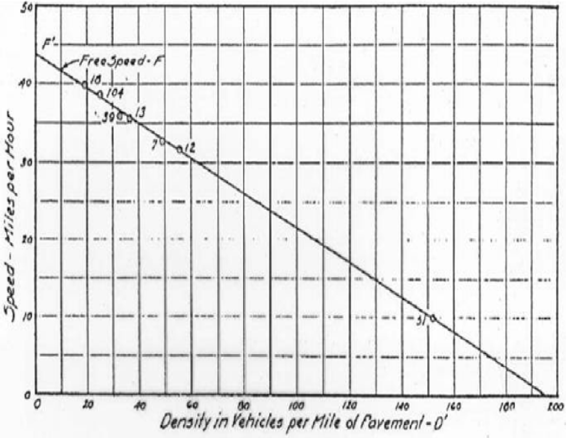
Effective 1d quantities



- **1d Density** $\rho^{\text{1d}}(x,t) = \int_{y=-W/w}^{W/2} \rho(x,y,t)\ \mathrm{d}y \approx W\rho(\boldsymbol{x},t)$ [ped./m]
- **Total flow** $Q(x,t) = \int_{y=-W/w}^{W/2} J(x,y,t)\ \mathrm{d}y \approx WJ(\boldsymbol{x},t)$ [ped/s]
- **Local speed** $V(x,t) = Q(x,t)/\rho^{\text{1d}}(x,t)$ [m/s]

5.4. Traffic Stream Models

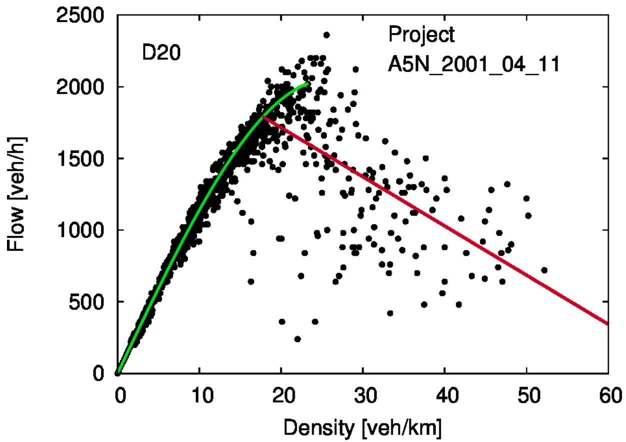
a **Traffic Stream Model** is just a fixed relation between two of the three basic macroscopic quantities local density ρ, flow Q, and local speed V.



Greenshield's relation: $V(\rho) = V_0 \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$

The early days of traffic data: Greenshields (1935)

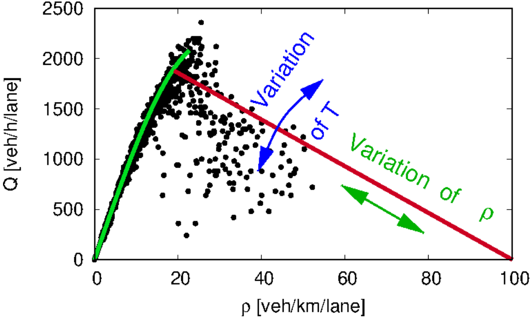
Flow-density data and fundamental diagram



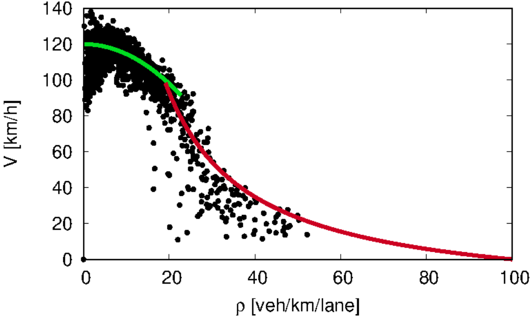
- The traffic-stream relation $Q(\rho)$ is called the **fundamental diagram**
- It can be estimated by flow-density data *taking care of the systematic errors*

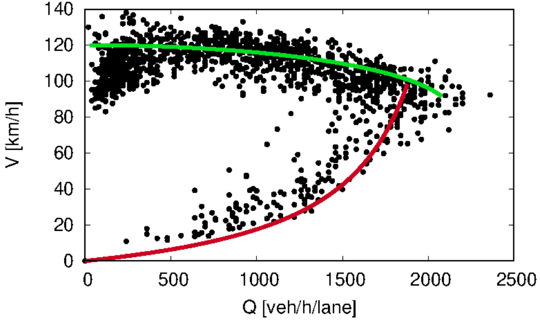
? How would the Greenshields fundamental diagram look like? $Q(\rho) = V_0 \, \rho \, \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$

"Two out of three" relations

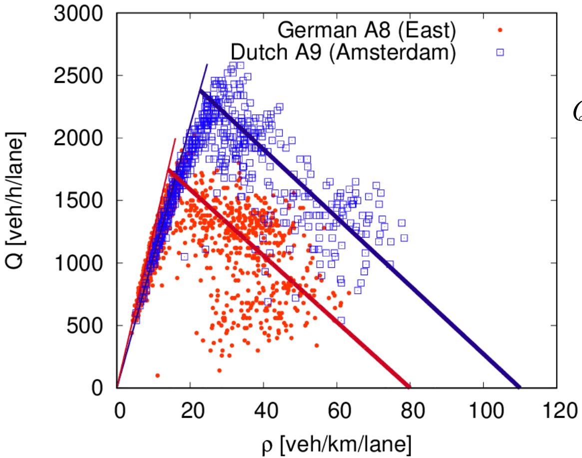


Together with the basic relation $Q = \rho V$, a single traffic stream relation fixes all three relations $Q(\rho)$, $V(\rho)$, and $Q(V)$





Triangular fundamental diagram (FD)



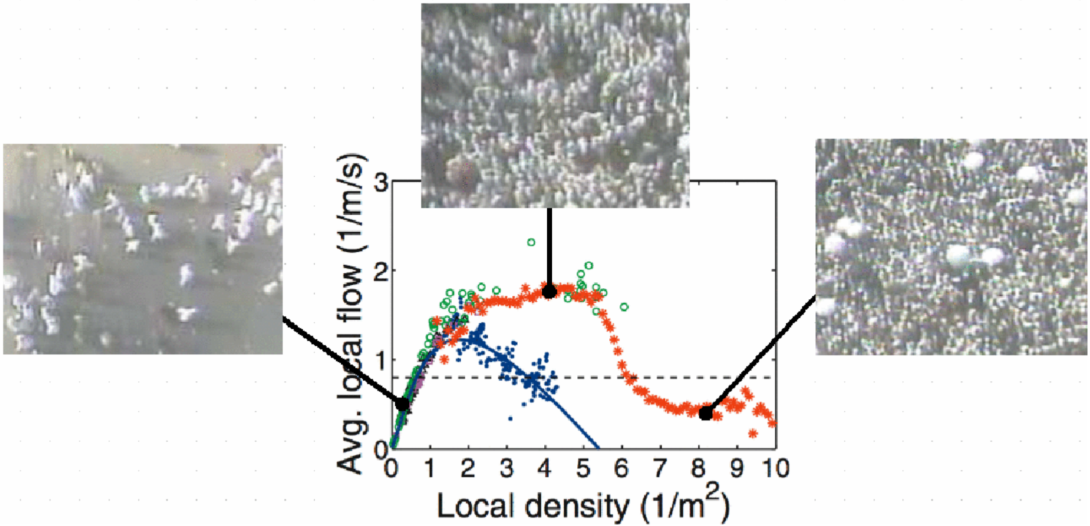
$$Q(\rho) = \min\left[V_0\rho, \quad \frac{1}{T}\left(1 - \frac{\rho}{\rho_{\text{max}}}\right)\right]$$

"free" branch "congested" branch

? Calculate the theoretical capacity and the density "at capacity"
$Q_{\text{max}} = V_0 \rho_c$ at $\rho_c = 1/(V_0 T + 1/\rho_{\text{max}})$

? Discuss the model parameters V_0, T, and ρ_{max}
V_0: desired speed, ρ_{max}: maximum density, T: Desired time gap following since gap $s = (1/\rho - 1/\rho_{\text{max}})$

Fundamental diagram for directed 2d traffic



Fundamental diagram for directed 2d traffic

Often, the simplest Greenshields FD for the flow density J as a function of the 2d density ρ is not too bad (only for fast pedestrians such as runners in sporting events, an asymmetric triangular fundamental diagram is better):

$$J(\rho) = V_0 \rho \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

Going from 2d to effective 1d:

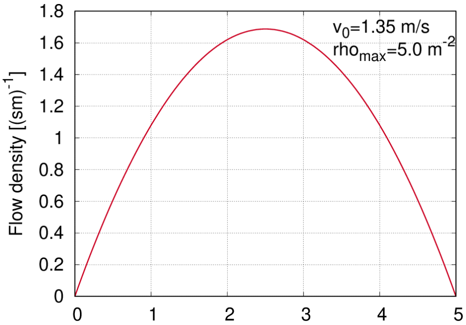
Assume a square grid for the pedestrian positions: longitudinal distance Δx_i=lateral "lane width" $\Delta W = \sqrt{1/\rho}$:

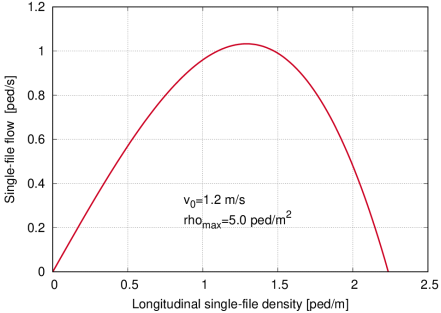
- several "single files" in parallel of width ΔW
- 1d-density of a single file: $\rho^{\text{1d}} = \rho \Delta W = \sqrt{\rho}$
- 1d-flow of this single file: $Q = J \Delta W = J/\sqrt{\rho} = J/\rho^{\text{1d}}$
- 1d-FD

$$Q(\rho^{\text{1d}}) = J((\rho^{\text{1d}})^2)/\rho^{\text{1d}} = V_0 \rho^{\text{1d}} \left(1 - \frac{(\rho^{\text{1d}})^2}{(\rho^{\text{1d}}_{\text{max}})^2}\right)$$

where $\rho^{\text{1d}}_{\text{max}} = \sqrt{\rho_{\text{max}}}$

Fundamental diagram for directed 2d traffic



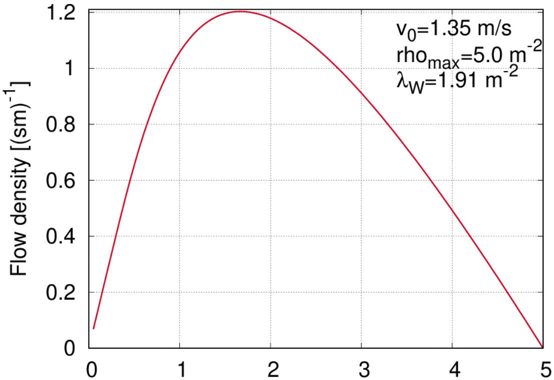


? Discuss the differences of the two FDs

? Give the capacity of a 30 m wide approach corridor assuming unidirectional pedestrian traffic flow and a Greenshields FD with parameters $V_0 = 1.2\,\mathrm{m/s}$ and $\rho_{\text{max}} = 5\,\mathrm{ped/m^2}$ (see the left image)

! Specific capacity $J_{\text{max}} = V_0 \rho_{\text{max}}/4 = 1.5\,\mathrm{ped/m/s}$, capacity $Q_{\text{max}} = W J_{\text{max}} = 45\,\mathrm{ped/s}$ or about 160,000 pedestrians per hour.

Weidmann FD

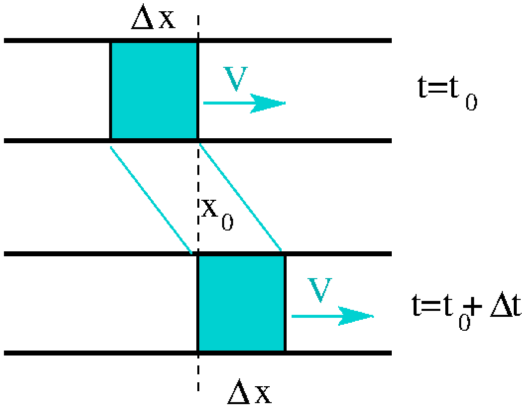


The popular Weidmann FD can be derived from microscopic social-force pedestrian flow models (→ Lecture 11). Its speed-density traffic stream relation reads (with the published parameter $\lambda = -1.913\,\mathrm{m}^{-2}$ and the same V_0 and ρ_{max})

$$J(\rho) = \rho V(\rho), \quad V(\rho) = V_0 \left\{ 1 - \exp\left[-\lambda \left(\frac{1}{\rho} - \frac{1}{\rho_{\mathsf{max}}} \right) \right] \right\}$$

In contrast to the greenshields FD, it is not symmetric

5.5. Hydrodynamic relation

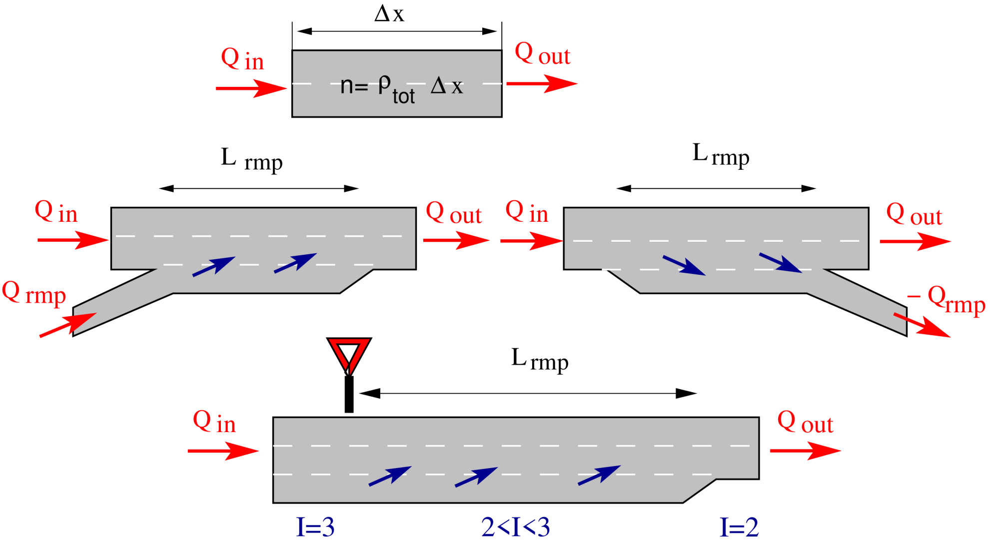


- Number of vehicles in blue-green box: $n \stackrel{\text{def}}{=} \rho \Delta x$
- Number of vehicles having passed x_0 during Δt: $n \stackrel{\text{def}}{=} Q \Delta t$
- hydrodynamic relation: $n = \rho \Delta x = Q \Delta t \ \Rightarrow$ $\frac{Q}{\rho} = \frac{\Delta x}{\Delta t} \stackrel{\text{def}}{=} V$

$Q = \rho V$ hydrodynamic relation

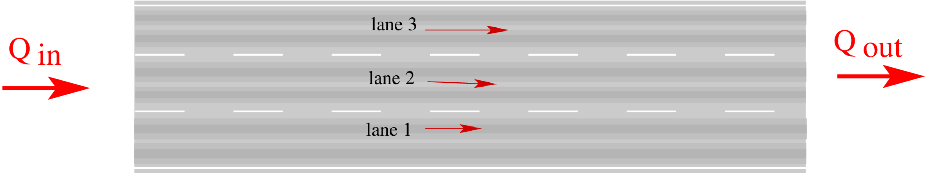
? Give the form for unidirectional 2d traffic. $J = \rho V$

5.6. Continuity Equation



The **continuity equation** just reflects vehicle/pedestrian conservation and is therefore *always* valid

Continuity equation along a homogeneous road



$$\frac{\mathrm{d}n}{\mathrm{d}t} = Q_{\text{in}} - Q_{\text{out}} = Q^{\text{tot}}(x,t) - Q^{\text{tot}}(x+\Delta x, t) \approx -\frac{\partial Q^{\text{tot}}}{\partial x}\Delta x$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(\int \rho^{\text{tot}}\,\mathrm{d}x\right) \approx \frac{\partial \rho^{\text{tot}}}{\partial t}\Delta x$$

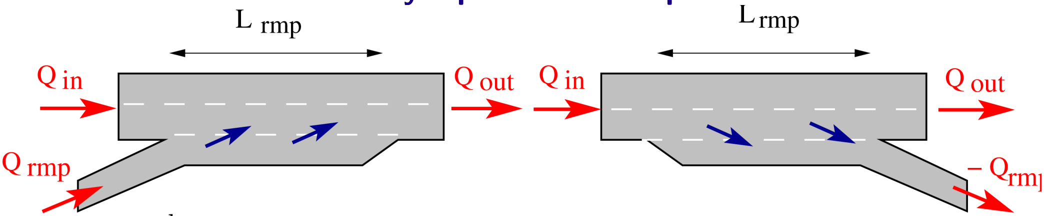
$\Rightarrow$ Total quantities: $$\frac{\partial \rho^{\text{tot}}}{\partial t} + \frac{\partial Q^{\text{tot}}}{\partial x} = 0$$

Effective quantities: $$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

Why is this continuity equation not valid for the lane quantities ρ_l, Q_l, V_l?

Because there are source terms due to lane changing

Continuity equation at ramp sections



$$\frac{\mathrm{d}n}{\mathrm{d}t} = Q_{\text{in}} - Q_{\text{out}} + Q_{\text{rmp}} = Q^{\text{tot}}(x,t) - Q^{\text{tot}}(x+L_{\text{rmp}},t) + Q_{\text{rmp}}$$
$$\approx -\frac{\partial Q^{\text{tot}}}{\partial x} L_{\text{rmp}} + Q_{\text{rmp}}$$

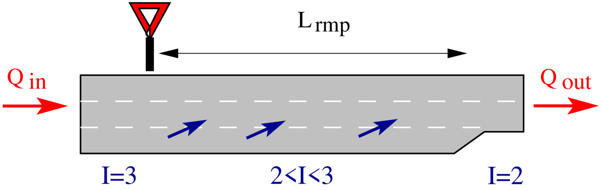
$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(\int \rho^{\text{tot}} \,\mathrm{d}x\right) \approx \frac{\partial \rho^{\text{tot}}}{\partial t} L_{\text{rmp}}$$

$$\frac{\partial \rho^{\text{tot}}}{\partial t} + \frac{\partial Q^{\text{tot}}}{\partial x} = \frac{Q_{\text{rmp}}}{L_{\text{rmp}}}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = \nu_{\text{rmp}}(x,t)$$

$$\nu_{\text{rmp}}(x,t) = \begin{cases} \frac{Q_{\text{rmp}}(t)}{L\, L_{\text{rmp}}} & x \text{ at merging/diverging zones} \\ 0 & \text{otherwise} \end{cases}$$

Continuity equation at changes of the lane number



- Variable effective lane number $L(x)$, here from $L = 3 \to 2$ along the merging zone of one or a few hundred meters:
- For the total quantities, the homogeneous continuity equation applies (why?): $\frac{\partial \rho^{\text{tot}}}{\partial t} + \frac{\partial Q^{\text{tot}}}{\partial x} = 0$
- For the effective quantities, we get

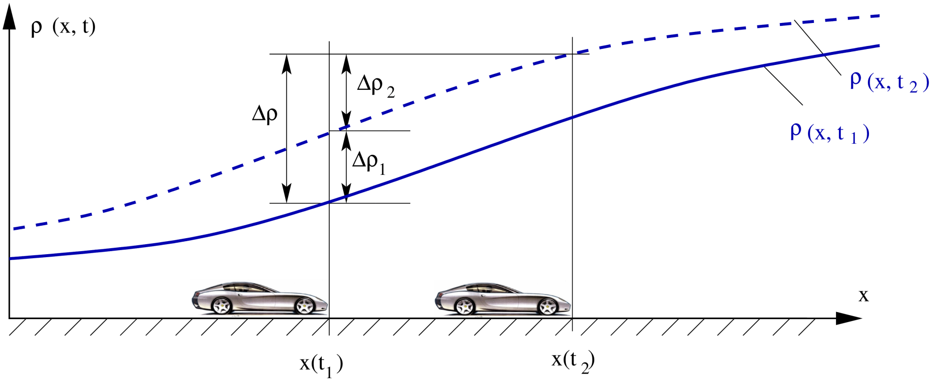
$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = -\frac{Q}{L}\frac{\mathrm{d}L}{\mathrm{d}x}$$

- The source terms of the ramp and lane-closing scenarios can be added

? Why use the more complicated effective continuity equation?

? Try to understand the lane-closing source in terms of the on-ramp source (and the lane opening in terms of an off-ramp)

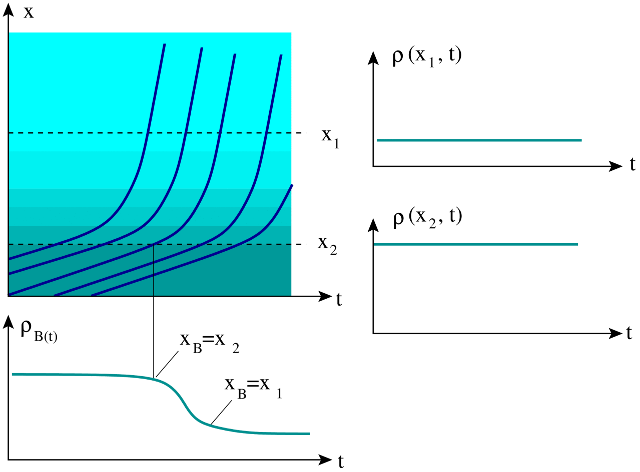
5.7. Coordinate Systems: Eulerian (Fixed Observer's) vs. Lagrangian (Driver's) View



Continuity equation from the floating car (driver's) perspective:

- Change of density: $\Delta\rho \approx \left(\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x}\right)\Delta t$ from the driver's perspective leads to the **total** or **convective** time derivative: $\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + V(x,t)\frac{\partial \rho}{\partial x}$
- Continuity equation in terms of the total derivative: $\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} = -\rho\frac{\partial V}{\partial x}$.
  Try to understand this intuitively!

Homogeneous, stationary, and steady state



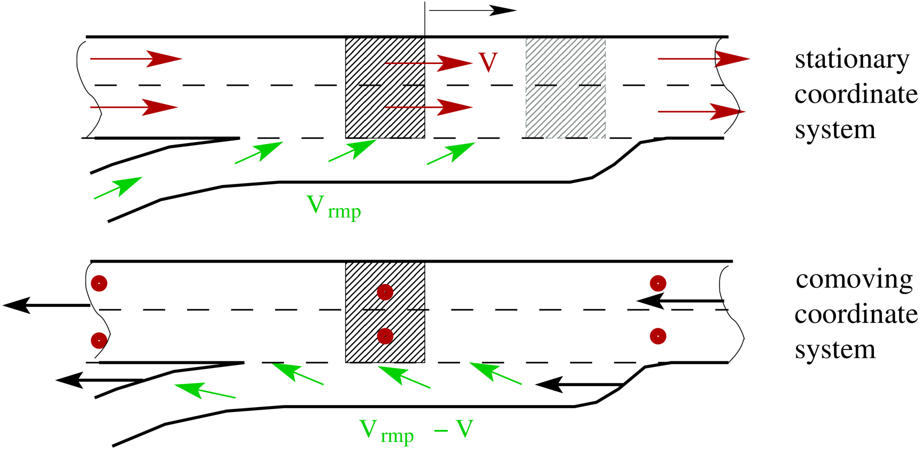
- Traffic flow is **homogeneous** if $\frac{\partial F}{\partial x} = 0$ where $F = \rho(x,t)$, $V(x,t)$, or any other macroscopic field as a function of x and t
- Traffic flow is **stationary** or in the **steady state** if $\frac{\partial F}{\partial t} = 0$

  Watch out: stationary != standing!

- Traffic flow is in the **homogeneous steady state** if $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial t} = \frac{\mathrm{d}F}{\mathrm{d}t} = 0$. This is assumed when formulating/deriving the *fundamental diagram*

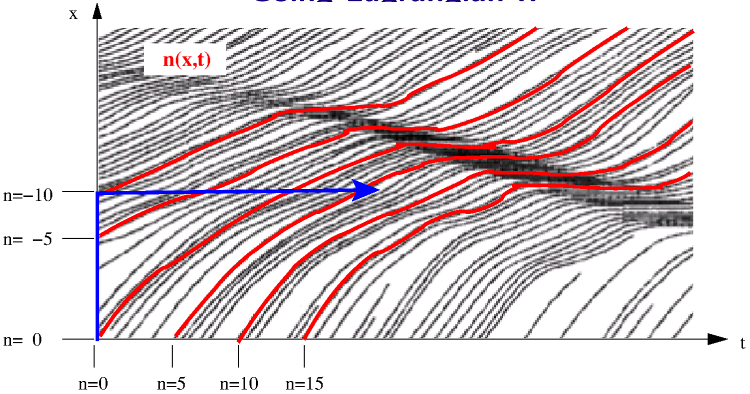
? Give examples of stationary nonhomogeneous and nonstationary homogeneous states

Going Lagrangian I



- Advantage: homogeneous systems become easier to describe since the convective term is eliminated
- Disadvantage: inhomogeneous systems become more complicated since ramps and other infrastructure stuff are moving

Going Lagrangian II



- Independent variable t: unchanged
- independent variable x: $\rightarrow$ real-valued **vehicle index** n (first vehicle has lowest index):

$$x \rightarrow n(x,t) = -\int_0^x \rho(x',0)\,\mathrm{d}x' + \int_0^t Q(x,t')\,\mathrm{d}t'$$

- dependent variables speed $V(x,t) \rightarrow v(n,t)$
- dependent variables density becomes **distance headway field** $\rho(x,t) \rightarrow 1/h(n,t)$ (name it h instead of d to avoid confusion with differential operators)

Lagrange Continuity equation for homogeneous roads: derivation

- Lagrangian variables: $\rho(x,t) = \frac{1}{h(n(x,t),t)}, \quad V(x,t) = v(n(x,t),t)$
- The definitions of flow and density directly give

$$\frac{\partial n}{\partial t} = Q = \rho V, \quad \frac{\partial n}{\partial x} = -\rho, \quad h = \frac{1}{\rho}, \quad \frac{\partial}{\partial x} = -\frac{1}{h}\frac{\partial}{\partial n}$$

- Transform the continuity equation (from the driver's view):

$$\begin{aligned}
0 &= \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\partial V}{\partial x} \\
&= \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\left[\frac{1}{h(n(x,t),t)}\right] - \frac{1}{h^2}\frac{\partial v}{\partial n} \\
&= -\frac{1}{h^2}\left(\frac{\partial h}{\partial n}\frac{\partial n}{\partial t} + \frac{\partial h}{\partial t} + V\frac{\partial h}{\partial n}\frac{\partial n}{\partial x}\right) - \frac{1}{h^2}\frac{\partial v}{\partial n} \\
&\overset{\frac{\partial n}{\partial t}=\rho V,\, \frac{\partial n}{\partial x}=-\rho}{=} -\frac{1}{h^2}\left(\rho V\frac{\partial h}{\partial n} + \frac{\partial h}{\partial t} - \rho V\frac{\partial h}{\partial n}\right) - \frac{1}{h^2}\frac{\partial v}{\partial n} \\
&= -\frac{1}{h^2}\left(\frac{\partial h}{\partial t} + \frac{\partial v}{\partial n}\right)
\end{aligned}$$

Lagrange Continuity equation for homogeneous roads: result

$$\frac{\partial h}{\partial t} + \frac{\partial v}{\partial n} = 0$$

Lagrange form of the continuity equation

- This result is plausible by integrating the second term over one unit of the index variable (because n is dimensionless, the lhs. is multiplied by one):

$$\frac{\partial h}{\partial t} + v(n+1,t) - v(n,t) = 0 \quad \Rightarrow \quad \frac{\partial h}{\partial t} = v_{\text{lead}} - v$$

h increases at a rate of the relative speed leader-follower.

? Why is the Lagrange form less efficient if there are bottlenecks?

Because bottlenecks are moving in this view

Problems

? Using the continuity equation, show that the total number of vehicles on a closed ring road with varying number of lanes $L(x)$ (but no on- or off-ramps) never changes.

! Integrate continuity equation for the total quantities over the circumference L:
$\int_{x=0}^{L} \left(\frac{\partial \rho^{\text{tot}}}{\partial t}\right) = -\int_{x=0}^{L} \left(\frac{\partial Q^{\text{tot}}}{\partial x}\right) = Q^{\text{tot}}(L) - Q^{\text{tot}}(0) = 0$

? How can we model the common behavior of drivers merging early onto the highway if there is free traffic and merging late (near the end of the ramp) in congested conditions?

! Change the constant ramp term $Q_{\text{rmp}}/L_{\text{rmp}} = wQ_{\text{rmp}}$ with $w = 1/L_{\text{rmp}} =$ const. to a variable $w(x)$ normalized to $\int_0^{L_{\text{rmp}}} w(x) = 1$

? Use the continuity equation to determine the traffic flow $Q(x)$ in a stationary state assuming a constant per-lane demand $Q(x, 0)$ and (iii) homogeneous road, (ii) ramps, (iii) a variable number of lanes.

! Stationarity means $\frac{\partial \rho}{\partial t} = 0$, so integrate over $\frac{\partial Q}{\partial x}$ plus source terms

? Consider a three-to-two lane closing and a constant inflow $Q_{\text{in}} = Q^{\text{tot}}(0, t) = 3,600\,\text{veh/h}$. Find the average per-lane density $\rho(x)$ and the average flow Q with respect to the two continuous lanes assuming a density-independent vehicle speed of 108 km/h (i.e., capacity $Q_{\text{max}} > 1,800\,\text{veh/h/lane}$) and a merging zone of length $L = 500\,\text{m}$. Compare with a continuous two-lane road with an on-ramp. Homework