

Traffic Flow Dynamics and Simulation

SS 2024, Solutions to Work Sheet 11, page 1

Solution to Problem 11.1: Pilgrimage in Mekka

- (a) *Density*: Pedestrians are moving in a 2d area, so only a definition of the density as # pedestrians per m^2 makes sense:

$$[\rho] = 1 \text{ m}^{-2}$$

(of course, in a single-file environment, we still have #per per meter) *Flow*: For a cross section (width) of the pedestrian passageway of more than about $W = 1.5 \text{ m}$, the flow [# pedestrians per second] at a given density increases proportional to the width W .¹ So, in this case, the flow Q [#ped/s] scales as

$$Q = Jw$$

where the proportionality factor denotes the additional flow per meter of cross section. Hence, it is appropriately named **flow density**

$$[J] = 1 \text{ ms}$$

J denotes the number of pedestrians per second and per meter of cross section.²

- (b) Typically, when observing pedestrian flow, one records a video from a fixed position (or a drone) and extracts trajectories $(x_i(t), y_i(t))$ for all pedestrians out of it. Because of this full microscopic information, systematic errors can be avoided by appropriate analysis/interpretation, e.g., as described in part (b).

Notice that for irregularly moving pedestrians (as on Christmas markets), only density but no flow or flow density makes sense

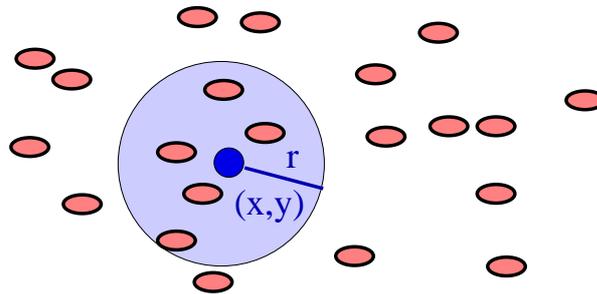
(c) Extracting the macroscopic quantities from trajectory data

- **Density** $\rho(x, y, t)$ Use a video still at time t and count all pedestrians that are inside a circle of radius r centered at (x, y) :

$$\rho(x, y, t) = \frac{\text{\#pedestrians in circle}}{\pi r^2} = \frac{\sum_i 1}{\pi r^2}$$

¹This is valid for unidirectional use; for bidirectional use, the minimum width is larger; for pedestrians moving in all directions, the concept *flow* is not well defined

²In lane-based vehicular traffic, this proportionality corresponds to the proportionality of the capacity with the number of lanes. In fact, one could define a flow density $J = Q/W_{\text{lane}}$ by dividing the flow by the lane width W_{lane} .



- * $r \approx 3$ m should be microscopically large (contains several pedestrians) and macroscopically small (smaller than the scale of the spatial structures to be resolved)
- * You could also use more refined kernels, e.g., cone-like or 2d Gaussian
- * Alternatively, you could define

$$\rho(x, y, t) = 1/A_{\text{voronoi}}$$

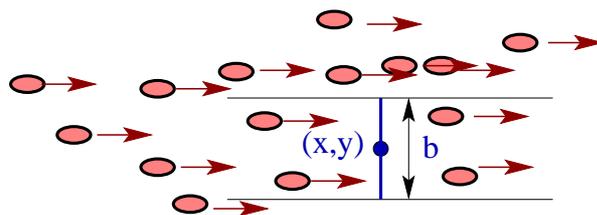
where A_{voronoi} is the area of the Voronoi cell containing the point (x, y)

- **Local space-mean speed/velocity** $V(x, y, t)$: just take the arithmetic average of the speed/velocity of the pedestrians inside the circle or the speed of the pedestrian in its Voronoi cell:

$$V(x, y, t) = \frac{\sum_i v_i}{\sum_i 1}$$

- **Flow density** $J(x, y, t)$: define a line of length $L < W$ perpendicular to the flow and centered at (x, y) and determine the #pedestrians crossing this line in the time interval $[t - \tau/2, t + \tau/2]$:

$$J(x, y, t) = \frac{\text{\#pedestrians crossing}}{L\tau}$$



L and τ should be microscopically large and macroscopically small, e.g., $L = 2$ m, $\tau = 5$ s.

(d) (i) **Approximate parameters of the triangular fundamental diagram**

$$J(\rho) = \min(\rho V_0, -(\rho_{\text{max}} - \rho)w)$$

European event (blue flow-density data):

- Maximum density by the intersection of the extrapolated flow-density points with the x axis:

$$\rho_{\max} = 5.5 \text{ ped/m}^2$$

(no bias expected since the true spatial density can be estimated from the data)

- Desired speed from the gradient of the flow-density data for very low densities:

$$V_0 = 1 \text{ m/s}$$

- Wave velocity w by the gradient of the points on the congested side:

$$w = -2 \text{ ms})^{-1} / \rho_{\max} = -0.35 \text{ m/s}$$

(Notice that the time gap T does not make sense in the 2d context. It would have the unit sm (second time meter) and the value (intersection of the congested branch with the y axis) of about $T = 2 \text{ sm}$. Hence, it is better to use the wave velocity w as the primary parameter.)

Hajj pilgrimage: A triangular FD does not make sense

(ii) Approximate parameters for the parabolic Greenshields fundamental diagram

$$J(\rho) = V_0 \rho \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

European event (blue flow-density data):

- Maximum density: A bit lower than for the triangular case, $\rho_{\max} \approx 5 \text{ ped/m}^2$
- Desired speed: Such that the observed specific capacity $J_{\max} = 1.5 \text{ ped/(sm)}$ is the same as the theoretical one,

$$J_{\max} = \frac{V_0 \rho_{\max}}{4} \quad \Rightarrow \quad V_0 = \frac{4 J_{\max}}{\rho_{\max}} = 1.5 \text{ m/s}$$

Hajj pilgrimage:

$$\begin{aligned} \rho_{\max} &= 10 \text{ m/s}^2, \\ V_0 &= \frac{4 J_{\max}}{\rho_{\max}} = \frac{8 / (\text{ms})}{10 \text{ m/s}^2} = 0.8 \text{ m/s} \end{aligned}$$

Notice that there is a discrepancy between this estimation and the gradient of the points for density to zero (about 1 m/s) which arises from the fact that neither the triangular nor the Greenshields FD fit the data.³

³These data look like there was a measuring artifact/a wrong estimation method for the high values or the density, or that the pedestrian flow entered in a new phase.

- (e) This can be best explained for the free-flow part, where the isotropic condition $\rho = \rho_{1d}^2$ gives the free-flow relation

$$Q = JL = \rho V_0 L = \rho_{1d}^2 V_0 L,$$

increasing *quadratically* with the 1d density. This strange result becomes plausible when realizing that, at very low densities, doubling the 1d density means a double flow within each single file *and* also the double number of single files.

In effect, a single file just needs the lateral space

$$W_{\text{ped}} = \frac{1}{\sqrt{\rho_{\text{max}}}},$$

so, for single-file dynamics, we effectively have

$$\rho = \rho_{1d} \frac{1}{W_{\text{ped}}}, \quad \rho_{\text{max}} = \rho_{1d,\text{max}} \frac{1}{W_{\text{ped}}}$$

and for the triangular FD the flow

$$Q_{\text{single}} = W_{\text{ped}} J = W_{\text{ped}} \min(\rho V_0, -(\rho_{\text{max}} - \rho)w) = \min(V_0 \rho_{1d}, -(\rho_{1d,\text{max}} - \rho)w)$$

with $\rho_{1d,\text{max}} = 1/W_{\text{ped}}$ and the value $\rho_{1d,\text{max}} = 2.3 \text{ ped/m}$ for the European event.

This results in an unchanged wave velocity w and the effective desired time gap for congested single files:

$$T = -\frac{1}{w \rho_{\text{max},1d}} \approx 1,2 \text{ s}$$

Remarkably, the implicit time gap for pedestrians is of the same order as that for cars in city or freeway traffic

Solution to Problem 11.2: Social Force Model

- (a)
- B : range of the interaction: If one is sufficiently far away from the target pedestrian (distance $|\vec{x}| \gg |\delta\vec{d}|$), the minor semi-axis b essentially becomes equal to the distance because, then, the ellipse with focal points a distance $\delta\vec{d}$ from each other and containing the point \vec{x} converges to a circle of radius $|\vec{x}| = b$. Therefore, the potential and the force decays by a factor of $1/e$ when the distance is increased by B . Experience tells us that a decay distance $B = 1$ m is plausible.
 - A : strength of the interaction. As a minimal condition, the interaction should prevent passing-throughs of pedestrians (they are no ghosts!) requiring $v^2/2 < \Phi^{\text{int}}(0) = AB$ (cf problem Part (c)). Here, we have $AB = 2 \text{ m}^2/\text{s}^2$ preventing such collisions for speeds below $v_c = \sqrt{2AB} = 2 \text{ m/s} = 7.2 \text{ km/h}$. This should also be enough when considering that the minimum distance is about 0.5 m and also including the additional driving force of the desired speed (provided τ is not too small). Another plausibility test is the maximum deceleration given by A as well. This also makes a value $A = 2 \text{ m/s}^2$ plausible.
 - Δt : anticipation time horizon for the trajectory planning. As in car driving, it should be comparable to the reaction time and the time gap when following each other: OK
 - λ : anisotropy: The repulsive force is weakened by a factor λ when the target is not straight ahead but behind and partially weakened for other directions. Should be $\ll 1$, even zero is plausible: OK.
 - τ : free-flow adaptation time. Determines, together with \vec{v}_0 , the free-flow dynamics. Essentially measures the time a pedestrian needs to reach his/her desired walking speed from standstill which is plausible. Also gives the maximum free acceleration $v_0/\tau = 0.75 \text{ m/s}^2$ which is plausible as well. Finally, the maximum free-flow acceleration is lower than the maximum interaction acceleration of the order of $A = 2 \text{ m/s}^2$
- (b) Calculating the interaction potential essentially entails calculating the semi-axis $b(\vec{x})$. For straight-ahead approaches to a standing obstacle as in Situation I assuming $\vec{x} = (-r, 0)$, $\vec{v} = (v, 0)$, $r > 0$, and $r > v\Delta t$, the equation for b simplifies as follows:

$$\begin{aligned}
 b(r|v) &= \frac{1}{2} \sqrt{\left(|\vec{x}| + |\vec{x} + \Delta\vec{d}|\right)^2 - |\Delta\vec{d}|^2} \\
 &= \frac{1}{2} \sqrt{\left(r + |\vec{x} + \vec{v}\Delta t|\right)^2 - (v\Delta t)^2} \\
 &= \frac{1}{2} \sqrt{\left(r + (r - v\Delta t)\right)^2 - (v\Delta t)^2} \\
 &= \frac{1}{2} \sqrt{(2r - v\Delta t)^2 - (v\Delta t)^2} \\
 &= \frac{1}{2} \sqrt{4r^2 - 4rv\Delta t} &= \sqrt{r^2 - rv\Delta t}
 \end{aligned}$$

Inserting the values of Situation 1 gives with $\Delta t = 1$ s the semi-minor axis $b = 2.12$ m

and

$$\Phi^{\text{int}} = AB \exp(-b/B) = 0.240 \text{ m}^2/\text{s}^2$$

- (c) For a pedestrian moving along the x axis at position $x = -r < 0$ and a standing target pedestrian at the origin, we have $w(\phi) = 1$, so

$$\begin{aligned} \frac{dv}{dt} &= \frac{v_0 - v}{\tau} - \frac{d\Phi^{\text{int}}}{dx} \\ &= \frac{v_0 - v}{\tau} + Ae^{-b/B} \frac{db}{dx} \\ &= \frac{v_0 - v}{\tau} - Ae^{-b/B} \frac{db}{dr} \\ &= \frac{v_0 - v}{\tau} - Ae^{-b/B} \frac{r - v\Delta t/2}{b} \end{aligned}$$

For Situation I, we have $(v_0 - v)/\tau = 0$ (the pedestrian moves at its desired velocity), $b = 2.12 \text{ m}$ from above, $r = 3 \text{ m}$ and $v\Delta t/2 = 0.75 \text{ m}$, so

$$\frac{dv}{dt} = -0.254 \text{ m/s}^2$$

- (d) (i) If the potential depends only on \vec{x} and not on the speed (of the subject pedestrian) or explicitly on time (if the target pedestrian moves), the SFM acceleration for $\tau \rightarrow \infty$ and without the directionality $w(\phi)$ (e.g., for frontal approaches) reads

$$\frac{d\vec{v}}{dt} = -\nabla\Phi^{\text{int}}(\vec{x})$$

or for Situation I:

$$\frac{dv}{dt} = -\frac{d\Phi^{\text{int}}(x)}{dx}$$

Multiplying both sides with $v = \frac{dx}{dt}$ gives

$$\begin{aligned} v \frac{dv}{dt} &= -\frac{d\Phi^{\text{int}}(x)}{dx} \frac{dx}{dt} \\ \frac{d}{dt} \left(\frac{v^2}{2} \right) &= -\frac{d}{dt} (\Phi^{\text{int}}(x(t))) \quad \int dt \\ \frac{v^2}{2} &= -\Phi^{\text{int}}(x) + \text{const.} \end{aligned}$$

or

$$E = \frac{v^2}{2} + \Phi^{\text{int}}(x) = \text{const.}$$

- (ii) We calculate the constant energy from the initial conditions:

$$E = \Phi^{\text{int}} + \frac{1}{2}v^2 = 1.365 \text{ m}^2/\text{s}^2 < AB = 2 \text{ m}^2/\text{s}^2$$

This means, Pedestrian 1 stops before reaching the center of Pedestrian 2 ⁴

⁴with anticipation, there is even a larger margin but it is not possible to calculate it analytically.

- (iii) Without anticipation, we have $\Delta\vec{d} = \vec{0}$ and the equation for the semi-minor axis becomes (for a target pedestrian at the origin)

$$b = |\vec{x}| = r$$

For the stopped pedestrian, $v = 0$ but its energy is still the same as in the beginning, so we have

$$\begin{aligned} E &= \Phi^{\text{int}}(x) = AB e^{-x/B} \\ \ln\left(\frac{E}{AB}\right) &= -\frac{x}{B}, \end{aligned}$$

giving the stopping distance

$$x = x_s = -B \ln\left(\frac{E}{AB}\right) = 0.382 \text{ m.}$$

- (e) The target pedestrian is standing at the origin (like an unpenetrable obstacle). Since the subject pedestrian already has a negative y -component, he/she will surely swerve further to the right to avoid the target, i.e., the y -component of the acceleration should be negative.

Since the pedestrian walks presently at his/her desired speed, there is no free acceleration term. Further, we have $\phi = -20\pi/180 = -0.349$ and the directional strength dependence $w(\phi) = 0.972 > 0$ does not change the direction vector which is given solely by the sum of the unit vectors. Since $\vec{e}_{\vec{x}}$ does not have an y component, the y component of the acceleration is proportional to the y -component of $\vec{e}_{\vec{x}+\Delta\vec{d}}$. We have

$$\vec{e}_{\vec{x}+\Delta\vec{d}} = \frac{\vec{x} + (\vec{v} - \vec{v}_2)\Delta t}{|\vec{x} + (\vec{v} - \vec{v}_2)\Delta t|} = (-0.95, -0.306)$$

confirming the expectation. In fact, we have a negative y component if $v_y - v_{2y} < 0$

Calculating the full interaction acceleration (with $w = 1$) gives

$$-\nabla\Phi^{\text{int}}(\vec{x}) = (-0.226, -0.035) \text{ m/s}^2$$

- (f) We expect now the subject pedestrian to swerve to the left because this pedestrian anticipates that, after the anticipation time, the target pedestrian will be at his/her right side, so we expect a positive y acceleration. Such an anticipation is also contained in the SFM with the presented **elliptical specification II**. A calculation as in (e) gives, indeed,

$$-\nabla\Phi^{\text{int}}(\vec{x}) = (-0.227, 0.034) \text{ m/s}^2$$

confirming the expectation

Solution to Problem 11.3: Single-file fundamental diagram

see [lecture slides](#)