

## Traffic Flow Dynamics and Simulation

### SS 2024, Solutions to Work Sheet 9, page 1

#### Solution to Problem 9.1: Dynamics of a single vehicle between two signalized intersections

- (a) *parameters.* The free acceleration is the same as that of the OVM. Hence,  $v_0$  is the desired speed and  $\tau$  the adaptation time. If the model decelerates, it does so with the deceleration  $b$ . Since, at this deceleration, the kinematic braking distance to a complete stop is given by  $\Delta x_{\text{brake}} = v^2/(2b)$ , the vehicle stops at a distance  $s_0$  to the (stopping line of) the red traffic light. This explains the meaning of the last parameter. Notice that, in this model, vehicles would follow any leading vehicle driving at a constant speed  $v_l < v_0$  at the same gap  $s_0$ , i.e., the model does not include a safe gap. Nor does it contain a reaction time. The model is accident-free with respect to stationary obstacles, but not when slower vehicles are involved.
- (b) *free acceleration phase.* Here, the first condition of the model applies, so we have to solve the ordinary differential equation (ODE) for the speed

$$\frac{dv}{dt} = \frac{v_0 - v}{\tau} \quad \text{with } v(0) = 0.$$

The exponential ansatz  $e^{\gamma t}$  for the homogeneous part  $\frac{dv}{dt} = -v/\tau$  gives the solvability condition  $\gamma = 1/\tau$ . Furthermore, the general solution for the full inhomogeneous (ODE) reads

$$v(t) = Ae^{-t/\tau} + B.$$

The asymptotic  $v(\infty) = B = v_0$  yields the inhomogeneous part  $B$ . Determining the integration constant  $A$  by the initial condition  $v(0) = A + B = A + v_0 = 0$  gives  $A = -v_0$ , so the speed profile reads

$$v(t) = v_0 \left(1 - e^{-\frac{t}{\tau}}\right).$$

Once  $v(t)$  is known, we determine the trajectory  $x(t)$  by integrating over time. With  $x(0) = 0$ , we obtain

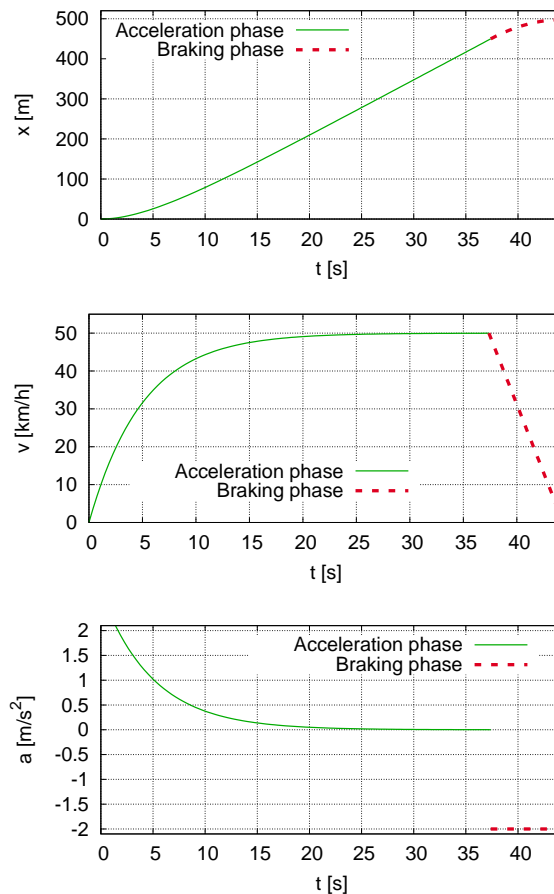
$$\begin{aligned} x(t) &= \int_0^t v(t') \, dt' = v_0 \int_0^t \left(1 - e^{-\frac{t'}{\tau}}\right) \, dt' \\ &= v_0 \left[ t' + \tau e^{-\frac{t'}{\tau}} \right]_{t'=0}^{t'=t} = v_0 t + v_0 \tau \left( e^{-\frac{t}{\tau}} - 1 \right). \end{aligned}$$

By identifying parts of this expression with  $v(t)$ , this simplifies to

$$x(t) = v_0 t - v(t)\tau.$$

Finally, to obtain the acceleration profile, we either differentiate  $v(t)$ , or insert  $v(t)$  into the right-hand side of the ODE. In either case, the result is

$$\dot{v} = \frac{v_0 - v}{\tau} = \frac{v_0}{\tau} e^{-t/\tau}.$$



- (c) *braking phase.* The red traffic light represents a standing virtual vehicle of zero length at the stopping line, so  $\Delta v = v$ . This phase starts at a distance

$$s_c = s_0 + \frac{v^2}{2b} = 50.2 \text{ m}$$

to the stopping line, and the vehicle stops at a distance  $s_0$  to this line.

- (d) *trajectory.* For the accelerating phase, the trajectory has already been calculated. The deceleration phase begins at the location

$$x_c = L - s_c = L - s_0 - \frac{v^2}{2b} \approx 450 \text{ m}.$$

To approximatively determine the time  $t_c$  at which the deceleration phase begins, we set  $v(t_c) = v_0$  to obtain

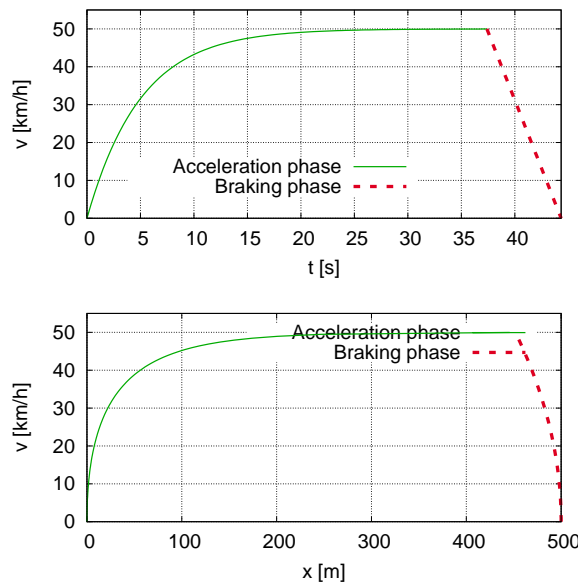
$$x_c(t_c) = v_0 t_c - v(t_c)\tau \approx v_0(t_c - \tau) \Rightarrow t_c = \frac{x_c}{v_0} + \tau = 32.4 \text{ s} + 5.0 \text{ s} = 37.4 \text{ s}.$$

With the braking time  $v_0/b$ , this also gives the stopping time

$$t_{\text{stop}} = t_c + \frac{v_0}{b} = 44.3 \text{ s}.$$

In summary, the speed profile  $v(t)$  can be expressed by (cf. the graphics below)

$$v(t) = \begin{cases} v_0 (1 - e^{-t/\tau}) & 0 \leq t < t_c, \\ v_0 - b(t - t_c) & t_c \leq t \leq t_{\text{stop}}, \\ 0 & \text{otherwise.} \end{cases}$$



## Solution to Problem 9.2: Full Velocity Difference Model

- (a) General plausibility arguments require the steady-state speed  $v_{\text{opt}}(s)$  to approach the desired speed  $v_0$  when the gap  $s$  tends to infinity. However, for an arbitrarily large distance to the red traffic light modeled by a standing virtual vehicle ( $\Delta v = v$ ), the FVDM vehicle accelerates according to

$$\dot{v} = \frac{v_0 - v}{\tau} - \gamma v = \frac{v_0}{\tau} - \left( \frac{1}{\tau} + \gamma \right) v.$$

- (i) From this it follows that the acceleration  $\dot{v}$  becomes zero for a terminal speed

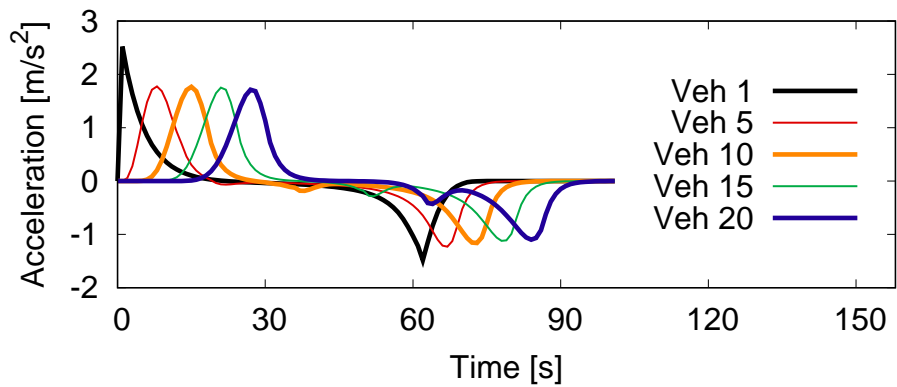
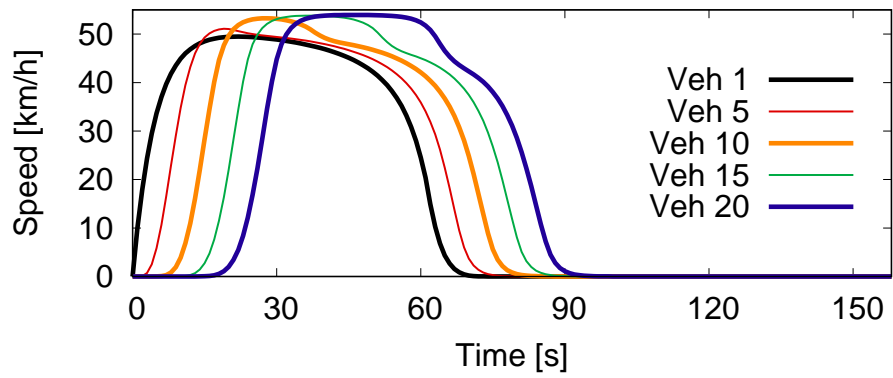
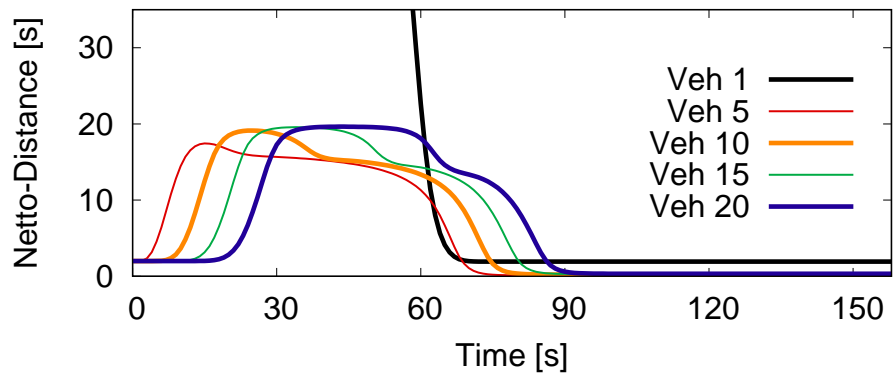
$$v^* = \frac{v_0}{1 + \gamma\tau}.$$

This is the maximum speed an initially standing FVDM vehicle can reach in this situation. It is significantly lower than  $v_0$ .

- (ii) For the parameter values of the problem statement,  $v^* = 13.5$  km/h agreeing with the simulated speed time series.
- (b) The “improved FVDM”

$$\frac{dv}{dt} = \frac{v_{\text{opt}}(s) - v}{\tau} - \gamma \frac{v - v_l}{\max[1, s/(v_0 T)]}$$

reverts to the normal OVM for gaps  $s \rightarrow \infty$ . Since the OVM does not show this inconsistency, the problem is solved while retaining the favourable property of the FVDM to include the relative speed  $v - v_l$  resembling much more a realistic and safer driving style. For reference, the situation of the problem statement is shown for the modified FVDM:



### Solution to Problem 9.3: Reaction to vehicles merging into the lane

(a) *Reaction for the IDM:* For  $v = v_0/2$ , the IDM steady-state space gap reads

$$s_e(v) = \frac{s_0 + vT}{\sqrt{1 - \left(\frac{v}{v_0}\right)^\delta}} = \frac{s_0 + \frac{v_0 T}{2}}{\sqrt{1 - \left(\frac{1}{2}\right)^\delta}}.$$

The prevailing contribution comes from the prescribed time headway (for  $s_0 = 2$  m and  $\delta = 4$ , the other contributions only make up about 10%). This problem assumes that the merging vehicle reduces the gap to the considered follower to half the steady-state gap,  $s = s_e/2 = v_0 T/4$ , while the speed difference remain zero. The new IDM acceleration of the follower (with  $a = 1$  m/s<sup>2</sup> and  $\delta = 4$ ) is therefore

$$\begin{aligned} \dot{v}_{\text{IDM}} &= a \left[ 1 - \left(\frac{v}{v_0}\right)^\delta - \left(\frac{s_0 + vT}{s}\right)^2 \right] \\ &\stackrel{(v=v_0/2, s=s_e/2)}{=} a \left[ 1 - \left(\frac{1}{2}\right)^\delta - \left(\frac{s_0 + v_0 T/2}{s_e/2}\right)^2 \right] \\ &\stackrel{s_e(v)=s_e(v_0/2)}{=} -3a \left[ 1 - \left(\frac{1}{2}\right)^\delta \right] = -\frac{45}{16} \text{ m/s}^2 = -2.81 \text{ m/s}^2. \end{aligned}$$

(b) *Reaction for the simplified Gipps’ model:* For this model, the steady-state gap in the car-following regime reads  $s_e(v) = v\Delta t$ . Again, at the time of merging, the merging vehicle has the same speed  $v_l = v = v_0/2$  as the follower, and the gap is half the steady-state gap,  $s = (v\Delta t)/2 = v_0\Delta t/4$ . The new speed of the follower is restricted by the safe speed  $v_{\text{safe}}$ :

$$v(t + \Delta t) = v_{\text{safe}} = -b\Delta t + \sqrt{b^2(\Delta t)^2 + v_l^2 + 2b(s - s_0)} = 19.07 \text{ m/s}.$$

This results in an effective acceleration

$$\left(\frac{dv}{dt}\right)_{\text{Gipps}} = \frac{v(t + \Delta t) - v(t)}{\Delta t} \approx -0.93 \text{ m/s}^2.$$

We conclude that the (simplified) Gipps’ model describes a more relaxed driver reaction compared to the IDM. Notice that both the IDM and Gipps’ model would generate significantly higher decelerations for the case of slower leading vehicles (dangerous situation).

### Solution to Problem 9.4: Reactions to a traffic light turning red

At the beginning, we have the dynamic state variables

$$s = 50 \text{ m}, \quad v = 15 \text{ m/s}, \quad \Delta v = v - v_l = 15 \text{ m/s}$$

(a) *Reaction for the IDM:* The dynamic desired gap is given by

$$s^* = s_0 + vT + \frac{v^2}{2\sqrt{ab}} \approx 96.5 \text{ m.}$$

The IDM acceleration term is zero since the driver cruises at his/her desired speed. The braking term and thus the full IDM deceleration calculates to

$$\dot{v}_{\text{IDM}} = -a \left( \frac{s^*}{s} \right)^2 = \underline{\underline{-3.73 \text{ m/s}^2}}.$$

(b) *Reaction for the simplified Gipps model:* We have

$$v_{\text{safe}} = -b\Delta t + \sqrt{b^2(\Delta t)^2 + 2b(s - s_0)} = 12.3 \text{ m/s.}$$

So, initially,

$$\dot{v}_{\text{Gipps}} = \frac{v_{\text{safe}} - v}{\Delta t} = \underline{\underline{-2.72 \text{ m/s}^2}}.$$

and  $\dot{v} = -b$  after the first  $\delta t$ -interval ensuring (as in the IDM) a safe stopping at a distance  $s_0$  before the stopping line.

Following plots show the IDM controlled braking maneuvers of the subject vehicle and a further follower initially following at a gap of 20 m.

