Methods in Transportation Econometrics and Statistics (Master)

Winter semester 2021/22, Solutions to Tutorial No. 9

Solution to Problem 9.1: Estimation of trivial and AC-only models

(a) In trivial models, we have $V_{ni} = 0$. For the binomial Probit model, this results in

$$P_1 = \Phi\left(\frac{V_1 - V_2}{\sqrt{2}}\right) = \Phi(0) = \frac{1}{2}, \quad P_2 = 1 - P_1 = \frac{1}{2},$$

and in the binomial Logit model to

$$P_i = \frac{e^0}{\sum_{i'=1}^{I} e^0} = \frac{1}{I} = \frac{1}{2}.$$

For the MNL, we have

$$P_i = \frac{\exp(V_i)}{\sum_{i'} \exp(V_{i'})} = \frac{1}{I}.$$

(b) For the AC-only model, the choice probabilities $P_{ni} = P_i$ do not depend on the decision maker. Hence, the log-likelihood in terms of the choice probabilities is given by

$$\tilde{L}(\vec{P}) = \sum_{n=1}^{N} \sum_{i=1}^{I} y_{ni} \ln P_i - = \sum_{i=1}^{I} N_i \ln P_i,$$

where $N_i = \sum_{n=1}^{N} y_{ni}$. Since the sum of the probabilities is equal to 1, we have an optimisation problem with one restraint (*note*: "s.t." is a standard abbreviation in math literature for "subject to"):

$$\sum_{i=1}^{I} N_i \ln P_i \stackrel{!}{=} \max, \text{ s.t. } \sum_{i=1}^{I} P_i = 1.$$

The general solution scheme for problems to maximize a general function $F(\vec{x})$ with one or more restraints is the following:

- Formulate all the restraints j in terms of functions $g_j(\vec{x}) = 0$
- Maximize the objective function augmented by *Lagrange mutlipliers* λ_j,

$$F(\vec{x}) - \sum_j \lambda_j g_j(\vec{x})$$

- Calculate the Lagrange multipliers by using the restraints.

Here, we have $\vec{x} = \vec{P}$, $F(\vec{P}) = \sum_I N_i \ln P_i$, and one restraint $g_1(\vec{P}) = g(\vec{P}) = \sum_i P_i - 1$. So,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}P_i}\left(\sum_{i'} N_{i'} \ln P_{i'} - \lambda(\sum_{i'} P_{i'} - 1)\right) &\stackrel{!}{=} 0, \\ \frac{N_i}{P_i} - \lambda &= 0 \Rightarrow P_i = \frac{N_i}{\lambda} \end{aligned}$$

Hence P_i is proportional to N_i and the restraint $\sum_{i'} P_{i'} = 1$ finally gives $P_i = N_i/N$.

Trivial mutinomial model with i.i.d. random utilities

Here, we have $P_i = P$ (this is not valid for correlated random utilities!) and the result comes directly from the restraint: $\sum_i P_i = IP = 1$, i.e., $P = 1/I$.

Solution to Problem 9.2: Considerations of a car salesman

(a) Revealed-choice since real buying decisions have been recorded.

(b)
- AC: δ_{i1}
- Socio-economic variables: age T_n of present car, offered discount R_n (whether the customer has accepted it or not), and the dummy variable whether the presently owned car had been bough as a new car.
- Characteristica: none

(c) The model would not be well specified since, as a socio-economic variable, the car age T_n does not depend on the alternatives, so, without the alternative-specific constant, there are no differences between the alternatives and hence, because of translation invariance, no effect.

(d) (i) Generally, we have

$$x_m^{\text{data}} = \sum_{n,i} x_{ni}^{(m)} y_{ni}, \quad x_m^{\text{mod}} = \sum_{n,i} x_{ni}^{(m)} P_{ni}(\vec{\beta}),$$

and in the current problem context

* Property sum X_1 related to $x_{ni}^{(1)} = \delta_{i1}$: Total number of successful deals (bought new cars)
* Property sum X_2 related to $x_{ni}^{(2)} = T_n\delta_{i1}$: Sum of the ages of the present cars from all customers who have actually bought a new car
* Property sum X_3 related to $x_{ni}^{(3)} = R_n\delta_{i1}$: Sum of the discounts offerend to customers who bought a new car

* Property sum X_4 related to $x_{ni}^{(4)} = \mathcal{N}_n \delta_{i1}$ where $\mathcal{N}_n = 1$ if the customer bought his/her last car as a new car, and zero otherwise: How many of the buyers bought their previous cars as a new car.

(ii) For $\vec{\beta} = 0$, all modelled choice properties are $P_i = 1/I = 1/2$, so we have for the realized and estimated property sums

$$
\begin{aligned}
&\text{Data: } X_1^{\text{data}} = \textstyle\sum_{n,i} \delta_{i1} y_{ni} = \sum_n y_{n1} = N_1 = 3, \quad \text{model } \vec{\beta} = 0 : X_1^{\text{mod}} = N/2 = 5\\
&\text{Data: } X_2^{\text{data}} = \textstyle\sum_{n,i} T_n \delta_{i1} y_{ni} = \sum_n T_n y_{n1} = 27, \quad \text{model } \vec{\beta} = 0 : X_2^{\text{mod}} = 75/2 = 37.5\\
&\text{Data: } X_3^{\text{data}} = \textstyle\sum_{n,i} R_n \delta_{i1} y_{ni} = \sum_n R_n y_{n1} = 8, \quad \text{model } \vec{\beta} = 0 : X_3^{\text{mod}} = 18/2 = 9\\
&\text{Data: } X_4^{\text{data}} = \textstyle\sum_{n,i} \mathcal{N}_n \delta_{i1} y_{ni} = \sum_n \mathcal{N}_n y_{n1} = 2, \quad \text{model } \vec{\beta} = 0 : X_4^{\text{mod}} = 5/2 = 2.5
\end{aligned}
$$

(e) Because it is unatractive to buy a new car if one already has a new car ($T_n = 0$), and no discount is offered ($R_n = 0$).

(f) In this situation, we have $T_n = 5$, $R_n = 2$ (2 000 € discount)Rabatt, and the "present car bough as a new car" dummy equals $\mathcal{N} = 1$. With the parameter estimator $\hat{\vec{\beta}} = (-9.2, 0.35, 2.2, 1.3)'$, we have

$$
\begin{aligned}
V_1 &= \hat{\beta}_1 + 5\hat{\beta}_2 + 2\hat{\beta}_3 + \hat{\beta}_4 = -1.75,\\
N &= e^{V_1} + e^{V_2} = e^{-1.75} + 1 = 1.174,\\
P_1 &= \frac{e^{V_1}}{N} = \underline{\underline{0.148}}
\end{aligned}
$$

(g) The variable "status of the present car" now can assume three values: (i) "bought as a new car", (ii) "bought as a used car", and (iii) "no car". This means, we now, in addition to $\mathcal{N}$ another dummy for the existence of an already owned car and the specification becomes

$$
V_{ni} = \beta_1 \delta_{i1} + \beta_2 T_n \delta_{i1} + \beta_3 R_n \delta_{i1} + \beta_4 \delta_{i1} \begin{cases} 1 & \text{last car was new} \\ 0 & \text{otherwise} \end{cases} + \beta_5 \delta_{i1} \begin{cases} 1 & \text{last car was used} \\ 0 & \text{otherwise.} \end{cases}
$$

Here,

- β_4 gives the relative propensity that customers whose last car was new make a deal compared to persons with no present car,
- β_5 gives the relative propensity of used-car owners to make a deal compared to persons with no present car.