Traffic Flow Dynamics and Simulation

SS 2024, Solutions to Work Sheet 8, page 1

Solution to Problem 8.1: Rules of thumb for the safe gap and braking distance

(a) In Continental Europe, the speedometer reading is in units of km/h, and the space gap is in units of meters. Since the gap increases with the speed, the *ratio* s/v, i.e., the time gap T is constant and given by (watch out for the units)

$$T = \frac{s}{v} = \frac{\frac{1}{2}\mathrm{m}\left(\frac{v}{\mathrm{km/h}}\right)}{v} = \frac{\frac{1}{2}\mathrm{m}}{\mathrm{km/h}} = \frac{0.5\,\mathrm{h}}{1\,000} = \frac{1\,800\,\mathrm{s}}{1\,000} = 1.8\,\mathrm{s}.$$

(b) One mile corresponds to 1.609 km. However, the US rule does not give explicit values for a vehicle length. Here, we assume $15\,\mathrm{ft} = 4.572\,\mathrm{m}$. In any case, the gap s increases linearly with the speed v, so the time gap $T = s/v$ is independent of speed. Implementing this rule, we obtain

$$T = \frac{s}{v} = \frac{\Delta s}{\Delta v} = \frac{15\,\mathrm{ft}}{10\,\mathrm{mph}} = \frac{4.572\,\mathrm{m}}{16.09\,\mathrm{km/h}} = \frac{4.572\,\mathrm{m}}{4.469\,\mathrm{m/s}} = 1.0\,\mathrm{s}.$$

Notice that, in the final result, we rounded off generously. After all, this is a rule of thumb and more significant digits would feign a non-existent precision.[1] Notice that this rule is consistent with typically observed gaps (see Lecture 02 on data).

(c) The kinematic *braking distance* is $s(v) = v^2/(2b)$, so the cited rule of thumb implies that the braking deceleration does not depend on speed. By inserting the kinematic braking distance into the rule, we obtain watch out for the units)

$$\begin{aligned} \frac{s}{\mathrm{m}} &= \frac{v^2}{100\,(\mathrm{km/h})^2} \\ s &= \frac{v^2\ 1\,\mathrm{m}}{100\,(\mathrm{km/h})^2} \stackrel{!}{=} \frac{v^2}{2b} \end{aligned}$$

Hence

$$b = \frac{100\,(\mathrm{km/h})^2}{2\,\mathrm{m}} = 3.86\,\mathrm{m/s}^2.$$

For reference, comfortable decelerations are below $2\,\mathrm{m/s}^2$ while emergency braking decelerations on dry roads with good grip conditions can be up to $10\,\mathrm{m/s}^2$, about $6\,\mathrm{m/s}^2$ for wet conditions, and less than $2\,\mathrm{m/s}^2$ for icy conditions. This means, the above rule could lead to accidents for icy conditions but is okay, otherwise.

[1] There is also a more conservative variant of this rule where one should leave one car length every five mph corresponding to the „two-second rule" $T = 2.0\,\mathrm{s}$.

Solution to Problem 8.2: A simple model for emergency braking maneuvers

(a) T_r = denotes the reaction time (the driver does nothing during this time), and b_{max} is the maximum deceleration (because it's an emergency).

(b) Assuming a fixed reaction timer T_r and a constant deceleration b_{max} in the braking phase, elementary kinematic relations yield following expressions for the braking and stopping distances $s_B(v)$ and $s_{\text{stop}}(v) = vT_r + s_B(v)$, respectively:

$$s_B(v) = \frac{v^2}{2b_{\text{max}}}, \quad s_{\text{stop}}(v) = vT_r + s_B(v)$$

with the numerical values

$$\begin{aligned} v &= 50\,\text{km/h}: \quad s_B(v) = 12.1\,\text{m}, \ s_{\text{stop}}(v) = 25.9\,\text{m}, \\ v &= 70\,\text{km/h}: \quad s_B(v) = 23.6\,\text{m}, \ s_{\text{stop}}(v) = 43.1\,\text{m}. \end{aligned}$$

(c) At first, we determine the initial distance such that a driver driving at $v_1 = 50\,\text{km/h}$ just manages to stop before hitting the child:

$$s(0) = s_{\text{stop}}(v_1) = 25.95\,\text{m}.$$

Now we consider a speed $v_2 = 70\,\text{km/h}$ but the same initial distance $s(0) = 25.95\,\text{m}$ as calculated above. At the end of the reaction time, the child is just

$$s(T_r) = s(0) - v_2 T_r = 6.50\,\text{m}$$

away from the front bumper. However, the driver would need the additional braking distance $v^2/(2b_{\text{max}}) = 23.6\,\text{m}$ for a complete stop. This results in a difference

$$s^* = \frac{v^2}{2b_{\text{max}}} - s(T_r) = 17.13\,\text{m}$$

With this information, the speed at collision can be calculated by solving $s^* = v^2/(2b_{\text{max}})$ for v, i.e.,

$$v_{\text{coll}} = \sqrt{2b_{\text{max}} s^*} = 16.56\,\text{m/s} = 59.6\,\text{km/h}.$$

Remark: Of course, in the theoretical test for the driver's license, you learn the answers by heart and tick 60 km/h ;-)

Solution to Problem 8.3: OVM acceleration on an empty road

(a) The maximum acceleration $a_{\max} = v_0/\tau$ is reached right at the beginning, $t = 0$.

(b) Prescribing $a_{\max} = 2\,\mathrm{m/s^2}$ and a desired speed $v_0 = 120\,\mathrm{km/h}$ determines the speed relaxation time by

$$\tau = \frac{v_0}{a_{\max}} = 16.7\,\mathrm{s}.$$

(c) Inserting the given ansatz (*cookbook rule: solution of an inhomogeneous linear ODE is the general solution of the homogeneous part plus a special solution of the full inhomogeneous equation*) into the equation of motion gives

$$-\frac{b}{\tau}e^{-t/\tau} = \frac{1}{\tau}\left(v_0 - a - be^{-t/\tau}\right)$$

In order to be an identity for all values of t, the prefactors of $e^{-t/\tau}$ and the constants must be equal simultaneously:

- prefactors of $e^{-t/\tau}$: $-b = -b$ (always satisfied)
- constants: $v_0 = a$

Furthernmore, the initial condition mus be satisfied:

$$v(0) = a + b = 0 \ \Rightarrow b = -a = -v_0,$$

so $a = v_0$ and $b = -v_0$ resulting in

$$v(t) = \underline{\underline{v_0\left(1 - e^{-\frac{t}{\tau}}\right)}}.$$

Remark: This differential equation can also be solved with another *cookbook rule* named **separation of variables** which can be used for any linear or nonlinear equation where the independent and dependent variables including their differentials can be "separated" on either side of the equation:

$$\begin{aligned}
\frac{dv}{v - v_0} &= -\frac{\mathrm{d}t}{\tau} \\
\ln(v - v_0) &= -\frac{t}{\tau} + C \\
v - v_0 &= e^{-\frac{t}{\tau}}e^C =: Ae^{-\frac{t}{\tau}} \\
v &= v_0 + Ae^{-\frac{t}{\tau}}
\end{aligned}$$

Determining the integration constant from the initial condition:

$$v(0) = v_0 + A = 0 \Rightarrow A = -v_0,$$

hence

$$\underline{\underline{v(t) = v_0(1 - e^{-\frac{t}{\tau}}).}}$$

(d) The acceleration can be obtained either by inserting $v(t)$ into the right-hand side of the OVM equation, or by directly taking the time derivative of $v(t)$:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v_0 - v}{\tau} = \frac{v_0}{\tau} e^{-\frac{t}{\tau}}.$$

(e) We require that, at a time t_{100} to be determined, the speed should reach the value $v_{100} = 100\,\mathrm{km/h}$:

$$v(t_{100}) = v_{100} = v_0 \left(1 - \mathrm{e}^{-\frac{t_{100}}{\tau}}\right).$$

Solving this condition for t_{100} gives

$$\frac{v_{100}}{v_0} = 1 - \mathrm{e}^{-\frac{t_{100}}{\tau}} \quad \Rightarrow \quad t_{100} = -\tau \ln\left(1 - \frac{100}{120}\right) \approx 29.9\,\mathrm{s}.$$