Traffic Flow Dynamics and Simulation

SS 2024, Solutions to Work Sheet 6, page 1

Solution to Problem 6.1: Jam propagation on a highway I: Accident

(a) The flow is restricted by the accident-induced bottleneck to the bottleneck capacity Q_B. Since one lane is closed, it is (in the best case) half the normal maximum total flow that can be read off the given fundamental diagram: $Q_\mathrm{B} = Q_{3,tot} = 1\,800\,\mathrm{veh/h}$.

(b) Since the demand/traffic flow at the time of the accident is greater than the bottleneck capacity, a traffic breadown with resulting congestion occurs. Directly from the conservation of the vehicle number it follows that the flow in the jammed region is also given by the bottleneck capacity: $Q_{2,tot} = Q_{3,tot} = Q_\mathrm{B} = 1\,800\,\mathrm{veh/h}$. However, the density must now be read off the *congested* branch of the fundamental diagram: $\rho_{2,tot} = \rho_\mathrm{cong}(Q_{2,tot}) = 120\,\mathrm{veh/km}$.

(c) Propagation velocity of the upstream jam front by the **shockwave formula**:

$$c_{12} = \frac{Q_{1,tot} - Q_{2,tot}}{\rho_{1,tot} - \rho_{2,tot}}.$$

So, we need also the traffic state at the inflow Region 1: $Q_{1,tot} = 2\,700\,\mathrm{veh/h}$, $\rho_{1,tot} = 30\,\mathrm{veh/km}$, so

$$v_\mathrm{g1} = \frac{2700/\mathrm{h} - 1800/\mathrm{h}}{30/\mathrm{km} - 120/\mathrm{km}} = -10\,\mathrm{km/h}.$$

(d) At 18:00 h, the demand suddenly becomes smaller than the bottleneck capacity[1] which, *simultaneously*, is also the flow on the congested side of the upstream jam front.[2] So, the demand suddenly is below the supply, so the queue shrinks (this is just as in a supermarket). So, the maximum jam length is given by

$$l_\mathrm{max} = -c_{12}\Delta t = 10\,\mathrm{km/h}\ 1\,\mathrm{h} = 10\,\mathrm{km}.$$

The maximum delay, of course, is imposed on the car just entering the jam at its maximum length. Hence, the *travel time* is given by

$$\tau_\mathrm{TT} = \frac{l_\mathrm{max}}{V_2} = \frac{l_\mathrm{max}\rho_{2,tot}}{Q_{2,tot}} = \frac{2}{3}\,\mathrm{h} = 40\,\mathrm{min}.$$

The *delay time* is the travel time minus the free-flow travel time:

$$\tau_\mathrm{delay} = \tau_\mathrm{TT} - \frac{l_\mathrm{max}}{V_0} = 6.67\,\mathrm{min}$$

where the free-flow speed is given by the slop of the free-flow branch of the fundamental diagram, $V_0 = 90\,\mathrm{km/h} = 1.5\,\mathrm{km/min}$.

[1] Here, we assume that the demand drop just reaches the jam front at 18:00 h.

[2] By definition, information travels upstream in congested regions, particularly the information on the bottleneck capacity.

(e) The dissolution time of the congestion is 18:00 h plus the time the congestion needs to shrink from 10 km to zero. This time is calculated using the new propagation velocity of the upstream front:

$$c'_{12} = \frac{900\,\text{veh/h} - 1\,800\,\text{veh/h}}{10\,\text{veh/km} - 120\,\text{veh/km}} = \frac{90}{11}\,\text{km/h}$$

So, the dissolution time is given by

$$t_{\text{diss}} - 18:00\,\text{h} = \frac{l_{\max}}{c'_{12}} = \frac{11}{9}\,\text{h}$$

which is about 19:13 h.

(f) Why the dissolution time is not 19 h, i.e., the growing and shrinking times of the congestion are the same? After all, the supply mismatch before 18:00 h is the same as the supply excess thereafter. The reason is that, after the complete dissolution, there are *less vehicles* in the 10 km section of the maximum jam extension than before the accident. The difference is

$$\Delta n_{\text{veh}} = (\rho_{1,tot} - \rho'_{1,tot}) l_{\max} = 200\,\text{vehicles}$$

These *additional vehicles* need a time $\Delta t = n_{\text{veh}}/900\,\text{veh/h} = 2/9\,\text{h}$ to vanish which concurs with the solution at (e). Mystery solved!

Solution to Problem 6.2: Jam propagation II: Uphill grade and lane drop

Subproblem (a):

As in the previous problem, we calculate the capacities with the capacity formula of the triangular fundamental diagram:

$$\begin{aligned} Q_{\max} &= \frac{V_0}{V_0 T + l_{\text{eff}}} = 2\,000\,\text{vehicles/h}, \\ Q^{\text{III}}_{\max} &= \frac{V_{03}}{V_{03} T_3 + l_{\text{eff}}} = 1\,440\,\text{vehicles/h}. \end{aligned}$$

Subproblem (b):

For the *total* quantities, lane drops, gradients, and other flow-conserving bottlenecks are irrelevant, and the continuity equation reads

$$\frac{\partial \rho_{\text{tot}}}{\partial t} + \frac{\partial Q_{\text{tot}}}{\partial x} = \frac{\partial Q_{\text{tot}}}{\partial x} = 0.$$

Since the inflow is constant, $Q_{\text{in}} = 2\,000\,\text{vehicles/h}$, and less the minimum capacity $C^{\text{III}} = 2Q^{\text{III}}_{\max} = 2\,880\,\text{vehicles/h}$, this amounts to stationary free traffic flow in all four regions I - IV with $Q_{\text{tot}} = Q_{\text{in}} = \text{const}$. From this information, we calculate the effective flow of all regions by dividing by the respective number of lanes, and the density by the free part of the fundamental diagram:

	V [km/h]	Q_{tot} [veh/h]	Q [veh/h/lane]	ρ_{tot} [veh/km]	ρ [veh/km/lane]
Region I	120	2 000	667	16.7	5.55
Region II	120	2 000	1 000	16.7	8.33
Region III	60	2 000	1 000	33.3	16.7
Region IV	120	2 000	1 000	16.7	8.33

Subproblem (c):

Traffic breaks down if the local traffic flow is greater than the local capacity. Thus, the jam forms at a location and at a time where and when this condition is violated, for the first time. Since the capacities in the four regions are given by 6 000, 4 000, 2 880, and 4 000 vehicles/h, respectively, the interface between regions II and III at $x = 3\,\text{km}$ is the first location where the local capacity can no longer meet the new demand $Q_{\text{in}} = 3\,600\,\text{vehicles/h}$. Traffic breaks down if the information of the increased demand reaches $x = 3\,\text{km}$. This information propagates through the regions I and II at

$$c_{\text{free}} = \frac{Q_q - Q_b}{\rho_a - \rho_b} = V_0 = 120\,\text{km/h} = 2\,\text{km/minute}$$

(where a and b stands for the state before and after the increased demand, respectively, in eiter region II or III). So,

$$x_{\text{breakd}} = 3\,\text{km}, \quad t_{\text{breakd}} = 16{:}01{:}30\,\text{h}.$$

Subproblem (d)

To determine density, flow, and speed of congested traffic in the regions I and II, we, again, adhere to the rule that free traffic flow is controlled by the upstream boundary while the total flow of congested regions and of regions downstream of „activated“ bottlenecks are equal to the bottleneck capacity at some earlier times determined by the information propagation velocities c_{free} and c_{cong}, respectively. Furthermore, densities inside congestions are calculated with the congested branch of the fundamental diagram while the free branch is used in all other cases. Denoting with regions Ib and IIb the congested sections of regions I and II, respectively, and with regions Ia and IIa the corresponding free-flow sections, this leads to following table for the traffic-flow variables:

	V [km/h]	Q_{tot} [veh/h]	Q [veh/h/lane]	ρ_{tot} [veh/km]	ρ [veh/km/lane]
Ia ($I = 3$)	16	3 600	1 200	30	10
Ib ($I = 3$)	120	2 880	960	180	60
IIa ($I = 2$)	36	3 600	1 800	30	15
IIb ($I = 2$)	120	2 880	1 440	80	40
III ($I = 2$)	60	2 880	1 440	48	24
IV ($I = 2$)	120	2 880	1 440	24	12

Notice that the local vehicle speed inside congested two-lane regions is more than *twice* that of three-lane regions.[3]

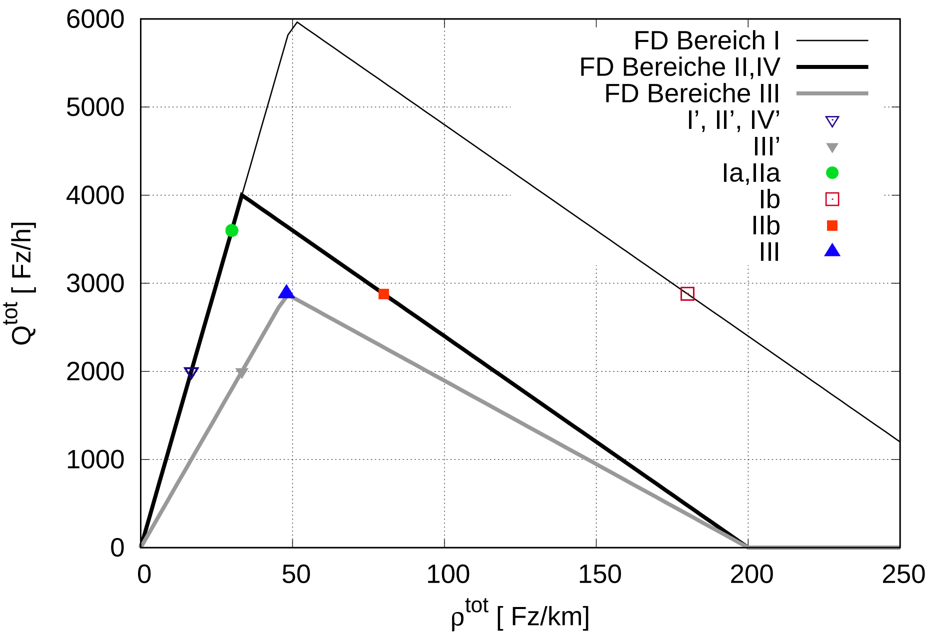
Subproblem (e)

To calculate the propagation velocities of the upstream jam front in the regions I and II, we use, again, the shock-wave formula together with the table of the previous subproblem:

$$v_g = \frac{\Delta Q}{\Delta \rho} = \begin{cases} -240/(60-10)\,\text{km/h} = -4.8\,\text{km/h} & \text{Interface Ia-Ib, situation (i)}, \\ -360/(40-15)\,\text{km/h} = -14.4\,\text{km/h} & \text{Interface IIa-IIb, situation (ii)}. \end{cases}$$

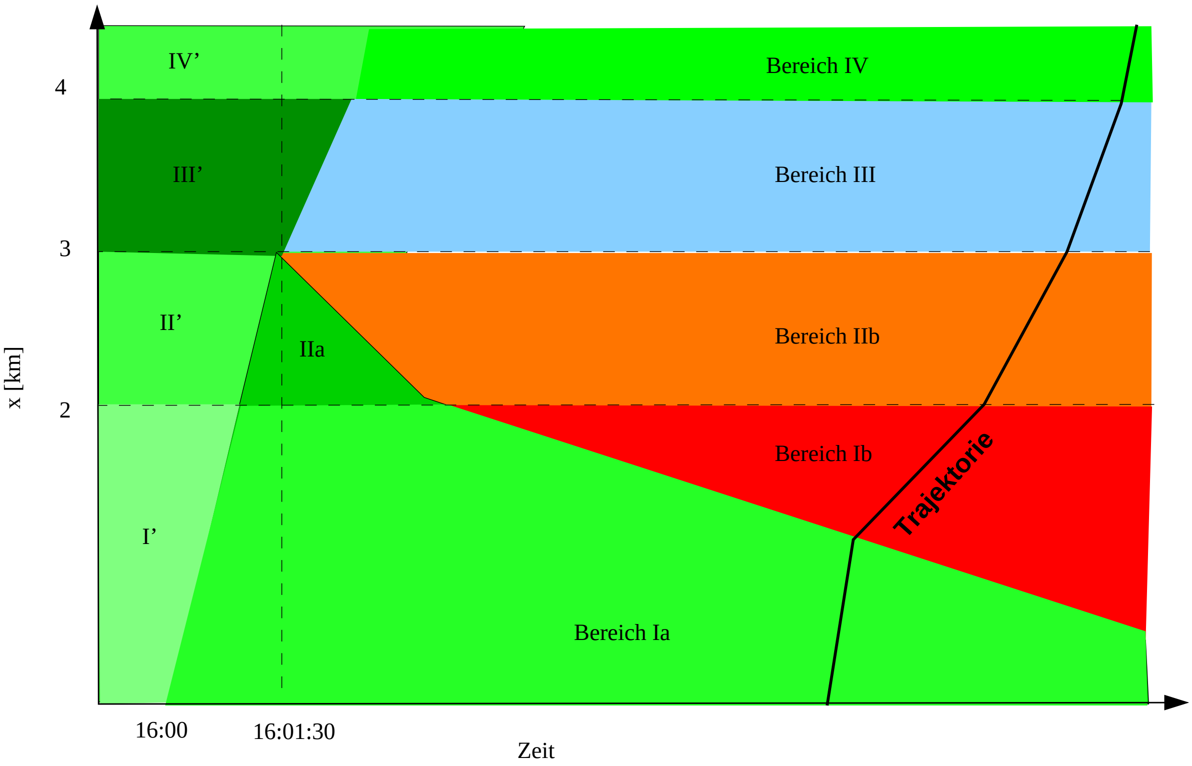
Subproblem (f)

The following figure depicts the fundamental diagrams $Q_{\text{tot}}(\rho_{\text{tot}})$ of the four regions and the traffic states. Notice that, in first-order macroscopic models, the state is *always* on the fundamental diagram.



The following figure depicts the spatiotemporal development of the congestion:

[3]When being stuck inside jams without knowing the cause, this allows to draw conclusions about the type of bottleneck, e.g., whether it is a three-to-two, or three-to-one lane drop.



Here, sections of the same *effective* (lane-averaged) densities are depicted with the same color:

- Region I': before the demand peak, $\rho = 5.55\,\mathrm{km}^{-1}$,
- Regione II', IV': before the demand peak: $\rho = 8.33\,\mathrm{km}^{-1}$,
- Region III': before the demand peak, $\rho = 16.7\,\mathrm{km}^{-1}$,
- Region Ia: demand peak, free, $\rho = 10\,\mathrm{km}^{-1}$,
- Region Ib: demand peak, congested, $\rho = 60\,\mathrm{km}^{-1}$,
- Region IIa: demand peak, free, $\rho = 15\,\mathrm{km}^{-1}$,
- Region IIa: demand peak, congested, $\rho = 40\,\mathrm{km}^{-1}$,
- Region III: maximum-flow state, $\rho = 24\,\mathrm{km}^{-1}$,
- Region IV: downstream of the congestion, $\rho = 12\,\mathrm{km}^{-1}$.

The propagation velocities of the transitions are as follows:

- free-free outside of the gradient region: $c_{\mathrm{I'Ia}} = c_{\mathrm{II'IIa}} = c_{\mathrm{IV'IV}} = V_0 = 120\,\mathrm{km/h}$,
- free-maximum-flow state within the gradient region: $c_{\mathrm{III'III}} = V_{0,\mathrm{III}} = 60\,\mathrm{km/h}$,

- upstream jam front Region II: $c_{\text{IIaIIb}} = -14.4\,\text{km/h}$,
- upstream jam front Region I: $c_{\text{IaIb}} = -4.8\,\text{km/h}$,
- downstream jam front (active bottleneck) at the begin of the gradient: $c_{\text{II, III}} = 0$ (the jam is “pinned” at the active bottleneck).

Subproblem (g)

Travel time for traversing the whole region when the vehicle enters the traffic jam at $x = x_c = 1\,\text{km}$ (i.e., Situation (i)):

$$\tau_{\text{TT}} = \frac{x_c}{V_0^{\text{I}}} + \frac{2\,\text{km} - x_c}{V_{\text{cong}}(60/\text{km})} + \frac{1\,\text{km}}{V_{\text{cong}}(40/\text{km})} + \frac{1\,\text{km}}{V_0^{\text{III}}} = 415\,\text{s} = 6\,\text{min},\ 55\,\text{s}.$$

with the congested speeds $V_{\text{cong}}(60\,\text{veh/km}) = 16\,\text{km/h}$ and $V_{\text{cong}}(40\,\text{veh/km}) = 36\,\text{km/h}$.

The first term denotes the episode with free traffic in Region I, the second on the time with congested trAffic in I, the next congested traffic in Region II, the fourth the time spent at the gradient (maximum-flow state at free-flow speed). According to the problem statement, the free-flow region downstream of the gradient is ignored.

Solution to Problem 6.3: Jam propagation III: Temporary partial road block

Subproblem (a):

With the values given in the problem statement, the capacity per lane reads

$$Q_{\text{max}} = \frac{V_0}{V_0 T + l_{\text{eff}}} = 2\,016\,\text{vehicles/h}.$$

The total capacity of the road in the considered driving direction without accident is just twice that value:

$$C = 2Q_{\text{max}} = 4\,032\,\text{vehicles/h}.$$

This exceeds the traffic demand 3 024 vehicles/h at the inflow ($x = 0$), so no jam forms before the accident, and only road section 1 exists. Since there are neither changes in the demand nor road-related changes, traffic flow is stationary and the flow per lane is constant:

$$Q_1 = \frac{Q_{\text{in}}}{2} = 1\,512\,\text{vehicles/h}, \quad V_1 = V_0 = 28\,\text{m/s}, \quad \rho_1 = \frac{Q_1}{V_0} = 15\,\text{vehicles/km}.$$

This also gives the travel time to traverse the $L = 10\,\text{km}$ long section:

$$t_{\text{trav}} = \frac{L}{V_0} = 357\,\text{s}.$$

Subproblem (b):

At the location of the accident, only one lane is open, so the bottleneck capacity

$$C_{\text{bottl}} = Q_{\text{max}} = 2\,016\,\text{vehicles/h}$$

does not meet the demand any more, and traffic breaks down at this location. This means, there are now three regions with different flow characteristics:

- Region 1, free traffic upstream of the congestion: Here, the situation is as in Subproblem (a).
- Region 2, congested traffic at and upstream of the bottleneck.
- Region 3, free traffic downstream of the bottleneck.

From the propagation and information velocities of perturbations in free and congested traffic flow, and from the fact that the flow but not the speed derives from a conserved quantity, we can deduce following general rules:

> *Free traffic flow is controlled by the flow at the upstream boundary, congested traffic flow and the traffic flow downstream of „activated“ bottlenecks is controlled by the bottleneck capacity.*

For the congested region 2 upstream of the accident (both lanes are available), this means

$$Q_2 = \frac{C_{\text{bottl}}}{2} = 1\,008\,\text{vehicles/h}.$$

To determine the traffic density, we invert the flow-density relation of the congested branch of the fundamental diagram,[4]

$$\rho_2 = \rho_{\text{cong}}(Q_2) = \frac{1 - Q_2 T}{l_{\text{eff}}} = 72.5\,\text{vehicles/km}.$$

Subproblem (c):

To calculate the propagation velocity of the shock (discontinuous transition free $\rightarrow$ congested traffic), we apply the shock-wave formula:

$$c^{\text{up}} = c_{12} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = -8.77\,\text{km/h}.$$

Subproblem (d):

After lifting the lane closure, the capacity is, again, given by $C = 2Q_{\text{max}} = 4\,032\,\text{vehicles/h}$, everywhere. In the LWR models, the outflow from congestions is equal to the local capacity, so the new outflow from the congestion is characterized by $Q_4 = C/2 = Q_{\text{max}}$, $V_4 = V_0$, and $\rho_4 = Q_4/V_0 = 20\,\text{vehicles/h}$. Furthermore, the transition from regions 2 to 3 (downstream jam front) starts to move upstream at a propagation velocity again calculated by the shock-wave formula:

$$c^{\text{down}} = c = c_{24} = \frac{Q_4 - Q_2}{\rho_4 - \rho_2} = -19.2\,\text{km/h}.$$

The jam dissolves if the upstream and downstream jam fronts meet. Defining t as the time past 15:00 h, x as in the figure of the problem statement, and denoting the duration of the bottleneck by $\tau_{\text{bottl}} = 30\,\text{minutes}$, we obtain following equations of motion for the fronts,

$$\begin{aligned} x_{\text{up}}(t) &= L + c^{\text{up}}\, t, \\ x_{\text{down}}(t) &= L + c(t - \tau_{\text{bottl}}). \end{aligned}$$

Setting these positions equal results in the time for complete jam dissolution:

$$t_{\text{dissolve}} = \tau_{\text{bottl}} \frac{c}{c - c^{\text{up}}} = 3\,312\,\text{s}.$$

The position of the last vehicle to be obstructed at obstruction time is equal to the location of the two jam fronts when they dissolve:

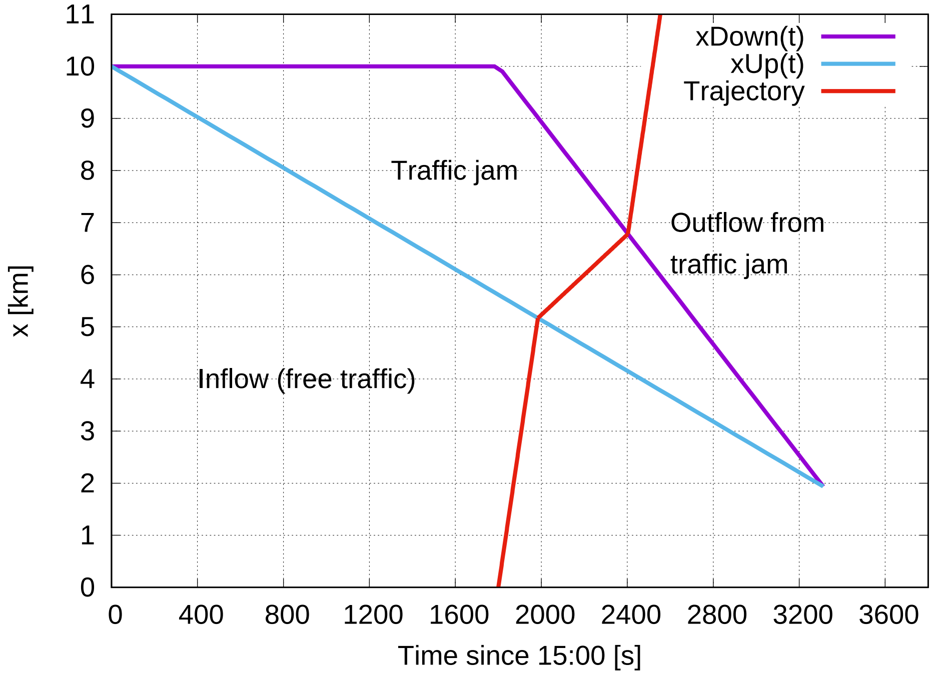
$$t_{\text{dissolve}} = L + c^{\text{up}}\, t_{\text{dissolve}} = 1\,936\,\text{m}.$$

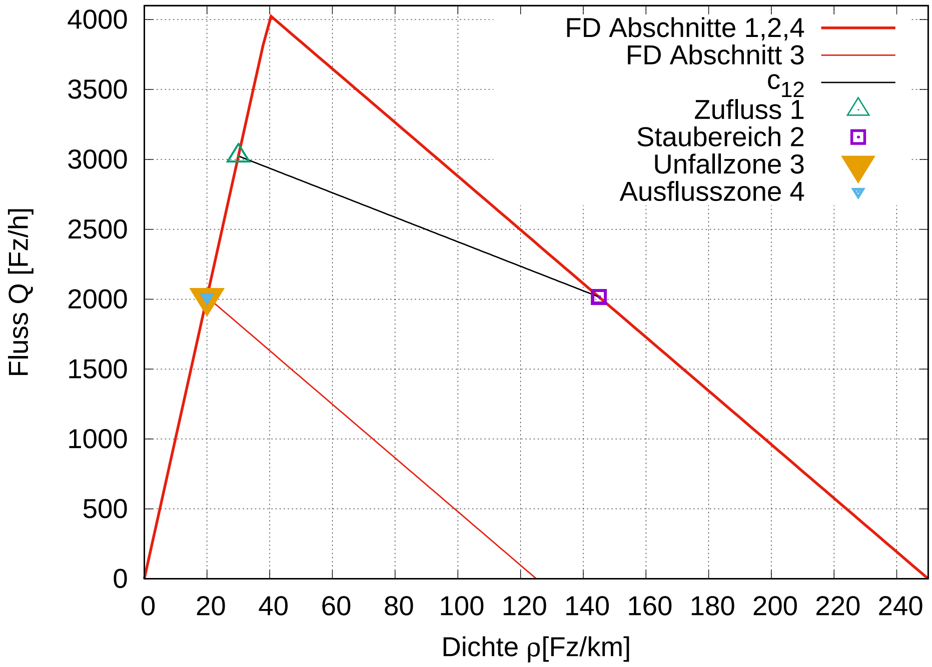
[4] *Beware:* The fundamental diagram and derived quantities (as $\rho_{\text{cong}}(Q)$) are always defined for the lane-averaged effective density and flow.

Subproblem (e):

In the spatiotemporal diagram, the congestion is restricted by three boundaries:

- Stationary downstream front at the bottleneck position $L = 10\,\text{km}$ for the times $t \in [0, \tau_{\text{bottl}}]$,
- Moving downstream front for $t \in [\tau_{\text{bottl}}], t_{\text{dissolve}}]$ whose position moves according to $x_{\text{down}}(t) = L + c(t - \tau_{\text{bottl}})$,
- Moving upstream front for $t \in [0, t_{\text{dissolve}}]$ whose position moves according to $x_{\text{up}}(t) = L + c^{\text{up}}\, t$





Subproblem (f):

We follow the vehicle trajectory starting at time $t = t_0 = 1\,800\,\mathrm{s}$ at the upstream boundary $x = 0$ by piecewise integrating it through the three regions (cf. the diagram):

(a) *Traversing the inflow region*: The vehicle moves at constant speed V_0 resulting in the trajectory $x(t) = V_0(t - t_0)$.

(b) *Traversing the jam*: To calculate the time t_{up} of entering the jam, we intersect the free-flow trajectory with the equations of motion $x_{\mathrm{up}}(t) = L + c^{\mathrm{up}}\, t$ for the upstream front:

$$t_{\mathrm{up}} = \frac{L + V_0 t_0}{V_0 - c^{\mathrm{up}}} = 1984\,\mathrm{s}.$$

The corresponding location $x_{\mathrm{up}} = V_0(t_{\mathrm{up}} - t_0) = 5168\,\mathrm{m}$. Hence, the trajectory reads

$$x(t) = x_{\mathrm{up}} + v_{\mathrm{cong}}(t - t_{\mathrm{up}}), \quad v_{\mathrm{cong}} = \frac{Q_2}{\rho_2} = 3.86\,\mathrm{m/s}.$$

(c) *Trajectory after leaving the jam*: Since, at time t_{up}, the bottleneck no longer exists, we calculate the exiting time by intersecting the trajectory calculated above with the equations of motion of the moving downstream front. This results in

$$\begin{aligned} t_{\mathrm{down}} &= \frac{L - x_{\mathrm{up}} - c t_0 + v_{\mathrm{cong}} t_{\mathrm{up}}}{v_{\mathrm{cong}} - c} = 2403\,\mathrm{s}, \\ x_{\mathrm{down}} &= x_{\mathrm{up}} + c_{\mathrm{cong}}(t_{\mathrm{down}} - t_{\mathrm{up}}) = 6\,783\,\mathrm{m}. \end{aligned}$$

After leaving the jam, the vehicle moves according to trajectory $x(t) = x_{\text{down}} + V_0(t - t_{\text{down}})$, so the vehicle crosses the location $x = L = 10\,\text{km}$ at time

$$t_{\text{end}} = t_{\text{down}} + \frac{L - x_{\text{down}}}{V_0} = 2\,518\,\text{s}.$$

In summary, we obtain for the total travel time to traverse the $L = 10\,\text{km}$ long section

$$\tau = t_{\text{end}} - t_0 = 718.1\,\text{s}.$$