Methods in Transportation Econometrics and Statistics (Master)

Winter semester 2021/22, Solutions to Tutorial No. 2

Solution to Problem 2.1: Matrix Rules

(a) For scalar products, a simple evaluation shows directly

$$\vec{a}'\vec{b} = \vec{b}'\vec{a} = \sum_i a_i b_i.$$

Notice the swapping of transpositions with the swapping of the position. In fact, this is a special case of a rule of transpositions on products to be discussed further below.

For general matrix products, commutativity does not apply, at least without additional transposition: If $\underline{\underline{A}}$ is a $n \times m$ matrix and B a $m \times k$ matrix, $\underline{\underline{AB}}$ is a $n \times k$ matrix while $\underline{\underline{BA}}$ is not even defined for $n \neq m$ or $m \neq k$. For square matrices $n = m = k$, we can check the 11-component:

$$(\underline{\underline{AB}})_{11} = \sum_k a_{1k} b_{k1}, \quad (\underline{\underline{BA}})_{11} = \sum_k b_{1k} a_{k1}.$$

In the first product, we need the matrix elements a_{11}, a_{12}, ... while we need a_{11}, a_{21},... for the second. This is not equivalent in case of non-symmetric matrices. Comparing the 12-components, it turns out that commutativity is not even valid for symmetric matrices.

(b) Both sides of the equation lead to the components

$$\left[(\underline{\underline{AB}})\underline{\underline{C}}\right]_{ij} = \left[\underline{\underline{A}}(\underline{\underline{BC}})\right]_{ij} = \sum_k \sum_l a_{ik} b_{kl} c_{lj},$$

so the associativity of the matrix products is reduced to the associativity of numbers (independence of which part of the product is calculated first).

(c) The relations $\underline{\underline{A}}(\vec{b} + \vec{c}) = \underline{\underline{A}}\vec{b} + \underline{\underline{A}}\vec{c}$ and $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{AB}} + \underline{\underline{AC}}$ follow directly from the elementwise addition of vector and matrix components and from the distributivity of numbersfolgen. A direct calculation makes this explicit.

(d) $\left(\underline{\underline{A}}'\right)' = \underline{\underline{A}}$ since the transposition swaps rows and columns and a double swap reverts the original expression.

(e) Again, we denote the j-th column of $\underline{\underline{A}}'$ (hence the j-th row of $\underline{\underline{A}}$) by $\vec{a}_j$. From this, it follows that

$$\left(\underline{\underline{A}}\vec{b}\right)' = \left[\begin{pmatrix} - & \vec{a}_1 & - \\ \cdots & \cdots & \cdots \\ - & \vec{a}_n & - \end{pmatrix} \cdot \begin{pmatrix} | \\ \vec{b} \\ | \end{pmatrix}\right]' = \begin{pmatrix} \vec{a}_1'\vec{b} \\ \vdots \\ \vec{a}_n'\vec{b} \end{pmatrix}' = \left(\vec{a}_1'\vec{b}, \ldots, \vec{a}_n'\vec{b}\right).$$

and

$$\vec{b}'\underline{\underline{A}}' = (-\vec{b}-) \cdot \begin{pmatrix} | & \vdots & | \\ \vec{a}_1 & \vdots & \vec{a}_n \\ | & \vdots & | \end{pmatrix} = \left(\vec{b}_1'\vec{a}, \ldots, \vec{b}_n'\vec{a}\right) = \left(\vec{a}_1'\vec{b}, \ldots, \vec{a}_n'\vec{b}\right).$$

Here, we have applied the multiplication rule "rows of the first object times columns of the second", symbolically indicated by horizontal and vertical points, respectively. The rule $\left(\underline{\underline{AB}}\right)' = \underline{\underline{B}}'\underline{\underline{A}}'$ can be derived in a similar way.

Alternatively, one can show this by explicit summation of components. For the object "matrix times vector transposed", we have for each component i i:

$$\left(\vec{b}'\underline{\underline{A}}'\right)_i = \sum_k b_k \left(\underline{\underline{A}}'\right)_{ki} = \sum_k b_k a_{ik} = \sum_k a_{ik} b_k = \left(\underline{\underline{A}}\vec{b}\right)_i'.$$

This means, the numerical values are identical. Furthermore, both products "transposed vector times matrix" and "(matrix times vector) transposed" result in a transposed vector. This means, also the type of mathematical object is identical.

For the object "matrix times matrix transposed", we obtain for each component ij:

$$\left(\underline{\underline{B}}'\underline{\underline{A}}'\right)_{ij} = \sum_k \left(\underline{\underline{B}}'\right)_{ik} \left(\underline{\underline{A}}'\right)_{kj} = \sum_k b_{ki} a_{jk} = \sum_k a_{jk} b_{ki} = \left(\underline{\underline{AB}}\right)_{ji} = \left(\underline{\underline{AB}}\right)_{ij}'.$$

(f) This identity can be shown by explicit calculation of the components. More elegantly, however, is applying the just derived transposition rule $(\underline{\underline{AB}})' = \underline{\underline{B}}'\underline{\underline{A}}'$ together with the switching property of the transposition onto $\underline{\underline{A}} = \underline{\underline{X}}$ and $\underline{\underline{B}} = \underline{\underline{X}}'$, respectively:

$$\left(\underline{\underline{X}}'\underline{\underline{X}}\right)' = \underline{\underline{X}}'\left(\underline{\underline{X}}'\right)' = \underline{\underline{X}}'\underline{\underline{X}}.$$

(g) The unit matrix $\underline{\underline{E}}$ (sometimes denoted by $\underline{\underline{1}}$) is a square matrix with ones on the diagonal, and zeroes, otherwise. It is the "neutral element" of the operation "matrix multiplication" and hence defines the inverse of square matrices. If this matrix is invertible or "regular" (notice that non-square matrices cannot be inverted in any case), following definition applies:[1]

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\underline{\underline{A}}^{-1} = \underline{\underline{E}}$$

Furthermore, we have, of course, $\underline{\underline{E}}' = \underline{\underline{E}}$ which means

$$\underline{\underline{E}}' = \left(\underline{\underline{A}}^{-1}\underline{\underline{A}}\right)' = \underline{\underline{A}}'\left(\underline{\underline{A}}^{-1}\right)'.$$

In the second line, we have used the transposition rule for matrix products derived earlier.

[1]Notice that multiplication with the unit matrix is the single exception from the non-commutativity of matrix products.

Multiplying this equation from left[2] by $\left(\underline{\underline{A}}'\right)^{-1}$, we obtain for the lhs.

$$\left(\underline{\underline{A}}'\right)^{-1}\underline{\underline{E}} = \left(\underline{\underline{A}}'\right)^{-1}.$$

The rhs. results in

$$\left(\underline{\underline{A}}'\right)^{-1}\left(\underline{\underline{A}}'\left(\underline{\underline{A}}^{-1}\right)'\right) = \left(\underline{\underline{A}}'\right)^{-1}\underline{\underline{A}}'\left(\underline{\underline{A}}^{-1}\right)' = \left(\underline{\underline{A}}^{-1}\right)',$$

or $\left(\underline{\underline{A}}'\right)^{-1} = \left(\underline{\underline{A}}^{-1}\right)'$, q.e.d. Notice that, in the first step of evaluating the rhs., we made use of the associativity of matrix multiplication.

Solution to Problem 2.2: Matrix Inversion

(a) The matrix multiplication cna be done efficiently "by hand" using *Falk's scheme* formalizing the multiplication rule

$$(\underline{\underline{AB}})_{ij} = i\text{-th row of } \underline{\underline{A}} \text{ times } j\text{-th column of } \underline{\underline{B}}.$$

For 2×2 matrices, this scheme reads

		e	f
		g	h
a	b	ae+bg	af+bh
c	d	ce+cg	cf+dh

. Here, the left factor (in the scheme at the upper left) equals $\underline{\underline{A}}^{-1}$, i.e.,

$$e = \frac{d}{\det\underline{\underline{A}}}, \quad f = -\frac{b}{\det\underline{\underline{A}}}, \quad g = -\frac{c}{\det\underline{\underline{A}}}, \quad h = \frac{a}{\det\underline{\underline{A}}}.$$

For example,

$$\left(\underline{\underline{AA}}^{-1}\right)_{11} = ae + bg = \frac{ad - bc}{ad - cb} = 1$$

The other three elements are tretaed similarly with the result

$$\underline{\underline{AA}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{E}},$$

q.e.d.

(b) Using Falk's scheme, we again prove the identity element by element:

[2] Because of the non-commutativity, one needs to specify if the matrix is multiplied "from left" or "from the right".

$$
\begin{array}{cccc|ccc}
 & & & & \mathrm{a} & \mathrm{b} & \mathrm{c} \\
 & & & & \mathrm{d} & \mathrm{e} & \mathrm{f} \\
 & & & & \mathrm{g} & \mathrm{h} & \mathrm{i} \\
\hline
 & \mathrm{ei\text{-}fh} & \mathrm{ch\text{-}bi} & \mathrm{bf\text{-}ce} & 1 & 0 & 0 \\
\frac{1}{aei+bfg+cdh-afh-bdi-ceg} & \mathrm{fg\text{-}di} & \mathrm{ai\text{-}cg} & \mathrm{cd\text{-}af} & 0 & 1 & 0 \\
 & \mathrm{dh\text{-}eg} & \mathrm{bg\text{-}ah} & \mathrm{ae\text{-}bd} & 0 & 0 & 1
\end{array}.
$$

For the 11 element of the product, we obtain

$$
\frac{a(ei-fh)+d(ch-bi)+g(bf-ce)}{aei+bfg+cdh-afh-bdi-ceg}=1 \qquad (1)
$$

The other elements are evaluated similarly.

Solution to Problem 2.3: Vector and Matrix Derivatives

Derivative $\frac{\partial}{\partial\vec{\beta}}\left(\vec{\beta}'\vec{a}\right)$:

$$
\frac{\partial}{\partial\vec{\beta}}\left(\vec{\beta}'\cdot\vec{a}\right)=\begin{pmatrix}\frac{\partial}{\partial\beta_0}\left(\sum_j\beta_j a_j\right)\\ \vdots \\ \frac{\partial}{\partial\beta_J}\left(\sum_j\beta_j a_j\right)\end{pmatrix}=\begin{pmatrix}a_0\\ \vdots\\ a_J\end{pmatrix}=\vec{a}
$$

Derivative $\frac{\partial}{\partial\vec{\beta}}\left(\vec{\beta}'\underline{\underline{A}}\vec{\beta}\right)$ taking into account the product rule of differentiation:

$$
\begin{aligned}
\frac{\partial}{\partial\vec{\beta}}\left(\vec{\beta}'\underline{\underline{A}}\vec{\beta}\right) &= \begin{pmatrix}\frac{\partial}{\partial\beta_0}\left(\sum_j\sum_k\beta_j A_{jk}\beta_k\right)\\ \vdots \\ \frac{\partial}{\partial\beta_J}\left(\sum_j\sum_k\beta_j A_{jk}\beta_k\right)\end{pmatrix}\\
&= \begin{pmatrix}\sum_k A_{0k}\beta_k+\sum_j\beta_j A_{j0}\\ \vdots \\ \sum_k A_{Jk}\beta_k+\sum_j\beta_j A_{jJ}\end{pmatrix}\\
&= \underline{\underline{A}}\vec{\beta}+\underline{\underline{A}}'\vec{\beta}.
\end{aligned}
$$