

# Methods in Transportation Econometrics and Statistics (Master)

Winter semester 2021/22, Tutorial No. 2

## Introduction: Vectors, Matrices, and Basic Operations on them

### (1) Vectors and Matrices

“Normal” vector = column vector  $\vec{a}$  with  $n$  components:

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{“}n \times 1\text{-matrix”}$$

row vector = transposed column vector:

$$\vec{a}' = (a_1, \dots, a_n) \quad \text{“}1 \times n\text{-matrix”}$$

$n \times m$ -matrix, i.e., a matrix with  $n$  rows and  $m$  columns.

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}. \quad \text{“}n \times m\text{-matrix”}$$

Transposed matrix: the rows and columns are swapped (the transposed vector above is a special case of that).

$$\underline{\underline{A}}' = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \dots & a_{nm} \end{pmatrix} \quad (\underline{\underline{A}}')_{ij} = a_{ji}.$$

Unit matrix  $\underline{\underline{E}}$  (neutral element with respect to matrix multiplication):

$$\underline{\underline{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Es gilt } \underline{\underline{A}} \cdot \underline{\underline{E}} = \underline{\underline{E}} \cdot \underline{\underline{A}} = \underline{\underline{A}}$$

Inverse  $\underline{\underline{A}}^{-1}$  of a regular (necessarily square) matrix  $\underline{\underline{A}}$ :

$$\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{A}} \cdot \underline{\underline{A}}^{-1} = \underline{\underline{E}}$$

(The only special case where a matrix product is commutative)

**(2) Additions and multiplications (the dots for the scalar and matrix products will be left out later on)**

Operation	Definition	Condition	Result
vector addition	$(\vec{a} + \vec{b})_i = a_i + b_i$	$n_a = n_b$	vector mit $n_a$ components
matrix addition	$(\underline{\underline{A}} + \underline{\underline{B}})_{ij} = a_{ij} + b_{ij}$	$n_A = n_B, m_A = m_B$	$n_A \times m_A$ -matrix
multiplication by a number	$(c\vec{a})_i = ca_i, (c\underline{\underline{A}})_{ij} = ca_{ij}$	none	vector or matrix
scalar produkt	$\vec{a}' \cdot \vec{b} = \vec{b}' \cdot \vec{a} = \sum_{i=1}^n a_i b_i$	$n_a = n_b$	Zahl (“Skalar”)
dyadic (tensor) product	$\vec{a} \cdot \vec{b}' = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_{n_b} \\ \vdots & & \vdots \\ a_{n_a} b_1 & \dots & a_{n_a} b_{n_b} \end{pmatrix}$	none	$n_a \times n_b$ - matrix
matrix times vector	$(\underline{\underline{A}} \cdot \vec{b})_i = \sum_{j=1}^m a_{ij} b_j$	$\underline{\underline{A}} = n \times m$ -matrix, $\vec{b} = m$ - vector	$n$ - vector
matrix-multiplikation	$(\underline{\underline{A}} \cdot \underline{\underline{B}})_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$	$\underline{\underline{A}} = n \times m$ -matrix, $\underline{\underline{B}} = m \times k$ - matrix	$n \times k$ -matrix

Notice that, formally, an  $n$ - vector is nothing else as a  $n \times 1$ -matrix, and a corresponding row vector a  $1 \times n$ -matrix. Furthermore, a number is a  $1 \times 1$ -matrix. Consequently, the rules for scalar and dyadic products, the multiplication rule for “matrix times vector”, and the addition and multiplication of normal numbers are just special cases of matrix multiplication!

### Problem 2.1: Matrix Rules

Prove by explicitly calculating the right-hand and left-hand sides of the following that the following statements and rules are valid:

- (a) commutativity is valid for scalar products with simultaneous transposition,  $\vec{a}'\vec{b} = \vec{b}'\vec{a}$ , but not for general (non-degenerated) matrix products:  $\underline{\underline{AB}} \neq \underline{\underline{BA}}$
- (b) Associativity for matrix products:  $\underline{\underline{AB}}\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{BC}})$
- (c) Distributivity for general matrix products such as  $\underline{\underline{A}}(\vec{b} + \vec{c}) = \underline{\underline{A}}\vec{b} + \underline{\underline{A}}\vec{c}$   
and  $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{AB}} + \underline{\underline{AC}}$
- (d) “Binary switching property” of the transposition operation:  $(\underline{\underline{A}}')' = \underline{\underline{A}}$
- (e) Rules for the transpose of vectors and matrices:  $(\underline{\underline{A}}\vec{b})' = \vec{b}'\underline{\underline{A}}'$  and  $(\underline{\underline{AB}})' = \underline{\underline{B}}'\underline{\underline{A}}'$
- (f) For arbitrary  $n \times m$  matrices  $\underline{\underline{X}}$ , the product  $\underline{\underline{X}}'\underline{\underline{X}}$  is a symmetric  $m \times m$  matrix:

$$(\underline{\underline{X}}'\underline{\underline{X}})_{ij} = (\underline{\underline{X}}'\underline{\underline{X}})_{ji}$$

- (g) For arbitrary regular (invertible) matrices, the operations of transposition and inversion are commutative, i.e.,  $(\underline{\underline{A}}')^{-1} = (\underline{\underline{A}}^{-1})'$ .

### Problem 2.2: Matrix Inversion

- (a) Given is a general  $2 \times 2$  Matrix

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove by means of matrix multiplication that the inverse of this matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \underline{\underline{A}}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det \underline{\underline{A}} = ad - bc \quad (1)$$

provided  $\underline{\underline{A}}$  is regular, i.e., the determinant  $ad - bc \neq 0$ .

- (b) (Exercise at home): Show by evaluating the matrix product  $\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1}$  that the inverse of regular  $3 \times 3$  matrices is given by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{aei + bfg + cdh - afh - bdi - ceg} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

### Problem 2.3: Vector and Matrix Derivatives

A vector derivative of a scalar function depending on a vector  $\vec{\beta}$  of variables is defined to be the column vector

$$\frac{\partial f(\vec{\beta})}{\partial \vec{\beta}} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial f}{\partial \beta_0} \\ \frac{\partial f}{\partial \beta_1} \\ \vdots \\ \frac{\partial f}{\partial \beta_J} \end{pmatrix}.$$

Apply this definition to the scalar functions  $f_1(\vec{\beta}) = \vec{\beta}'\vec{a}$  and  $f_2(\vec{\beta}) = \vec{\beta}'\underline{A}\vec{\beta}$  ( $\vec{a}$  and  $\underline{A}$  do not depend on  $\vec{\beta}$ ) and show that following derivation rules are valid:

$$\frac{\partial}{\partial \vec{\beta}} (\vec{\beta}'\vec{a}) = \frac{\partial}{\partial \vec{\beta}} (\vec{a}'\vec{\beta}) = \vec{a},$$

and

$$\frac{\partial}{\partial \vec{\beta}} (\vec{\beta}'\underline{A}\vec{\beta}) = (\underline{A} + \underline{A}')\vec{\beta}.$$