Methods in Transportation Econometrics and Statistics (Master)

Winter semester 2023/24, Tutorial No. 2

Introduction: Vectors, Matrices, and Basic Operations on them

(1) Vectors and Matrices

"Normal" vector = column vector $\vec{a}$ with n components:

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{“}n \times 1\text{-matrix”}$$

row vector = transposed column vector:

$$\vec{a}' = (a_1, \cdots, a_n) \qquad \text{“}1 \times n\text{-matrix”}$$

$n \times m$-matrix, i.e., a matrix with n rows und m columns.

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}. \qquad \text{“}n \times m\text{-matrix”}$$

Transposed matrix: the rows and columns are swapped (the transposed vector above is a special case of that).

$$\underline{\underline{A}}' = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \dots & a_{nm} \end{pmatrix} \qquad \left(\underline{\underline{A}}'\right)_{ij} = a_{ji}.$$

Unit matrix $\underline{\underline{E}}$ (neutral element with respect to matrix multiplication):

$$\underline{\underline{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad \underline{\underline{A}} \cdot \underline{\underline{E}} = \underline{\underline{E}} \cdot \underline{\underline{A}} = \underline{\underline{A}}$$

Inverse $\underline{\underline{A}}^{-1}$ of a regular (necessarily square) matrix $\underline{\underline{A}}$:

$$\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{A}} \cdot \underline{\underline{A}}^{-1} = \underline{\underline{E}}$$

(The only special case where a matrix product is commutative)

(2) Additions and multiplications (the dots for the scalar and matrix products will be left out later on)

Operation	Definition	Condition	Result
vector addition	$\left(\vec{a}+\vec{b}\right)_i = a_i + b_i$	$n_a = n_b$	vector mit n_a components
matrix addition	$\left(\underline{\underline{A}}+\underline{\underline{B}}\right)_{ij} = a_{ij} + b_{ij}$	$n_A = n_B,\ m_A = m_B$	$n_A \times m_A$-matrix
multiplication by a number	$(c\vec{a})_i = ca_i, \quad \left(c\underline{\underline{A}}\right)_{ij} = ca_{ij}$	none	vector or matrix
scalar produkt	$\vec{a}' \cdot \vec{b} = \vec{b}' \cdot \vec{a} = \sum\limits_{i=1}^{n} a_i b_i$	$n_a = n_b$	number ("scalar")
dyadic (tensor) product	$\vec{a} \cdot \vec{b}' = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_{n_b} \\ \vdots & & \vdots \\ a_{n_a} b_1 & \dots & a_{n_a} b_{n_b} \end{pmatrix}$	none	$n_a \times n_b$ - matrix
matrix times vector	$\left(\underline{\underline{A}} \cdot \vec{b}\right)_i = \sum\limits_{j=1}^{m} a_{ij} b_j$	$\underline{\underline{A}} = n \times m$-matrix, $\vec{b} = m$- vector	n- vector
matrix-multiplikation	$\left(\underline{\underline{A}} \cdot \underline{\underline{B}}\right)_{ij} = \sum\limits_{k=1}^{m} a_{ik} b_{kj}$	$\underline{\underline{A}} = n \times m$-matrix, $\underline{\underline{B}} = m \times k$ - matrix	$n \times k$-matrix

Notice that, formally, an n- vector is nothing else as a $n \times 1$-matrix, and a corresponding row vector a $1 \times n$-matrix. Furthermore, a number is a 1×1-matrix. Consequently, the rules for scalar and dyadic products, the multiplication rule for "matrix times vector", and the addition and multiplication of normal numbers are just special cases of matrix multiplikation!

Problem 2.1: Matrix Rules

Prove by explicitely calculating the right-hand and left-hand sides of the following that the following statements and rules are valid:

(a) commutativity is valid for scalar products with simultaneous transposition, $\vec{a}'\vec{b} = \vec{b}'\vec{a}$, but not for general (non-degenerated) matrix products: $\underline{\underline{AB}} \neq \underline{\underline{BA}}$

(b) Associativity for matrix products and matrix-vector products: $(\underline{\underline{AB}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{BC}})$, $(\vec{a}'\underline{\underline{B}})\underline{\underline{C}} = \vec{a}'(\underline{\underline{BC}})$, $(\underline{\underline{AB}})\vec{c} = \underline{\underline{A}}(\underline{\underline{B}}\vec{c})$, and the like.

(c) Distributivity for general matrix products such as $\underline{\underline{A}}(\vec{b} + \vec{c}) = \underline{\underline{A}}\vec{b} + \underline{\underline{A}}\vec{c}$ and $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{AB}} + \underline{\underline{AC}}$

(d) "Binary switching property" of the transposition operation: $\left(\underline{\underline{A}}'\right)' = \underline{\underline{A}}$

(e) Rules for the transpose of vectors and matrices: $\left(\underline{\underline{A}}\vec{b}\right)' = \vec{b}'\underline{\underline{A}}'$ and $\left(\underline{\underline{AB}}\right)' = \underline{\underline{B}}'\underline{\underline{A}}'$

(f) For arbitrary $n \times m$ matrices $\underline{\underline{X}}$, the product $\underline{\underline{X}}'\underline{\underline{X}}$ is a symmetric $m \times m$ matrix:

$$\left(\underline{\underline{X}}'\underline{\underline{X}}\right)_{ij} = \left(\underline{\underline{X}}'\underline{\underline{X}}\right)_{ji}$$

(g) For arbitrary regular (invertible) matrices, the operations of transposition and inversion are commutative, i.e., $\left(\underline{\underline{A}}'\right)^{-1} = \left(\underline{\underline{A}}^{-1}\right)'$.

Problem 2.2: Matrix Inversion

(a) Given is a general 2×2 Matrix

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove by means of matrix multiplication that the inverse of this matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \underline{\underline{A}}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det \underline{\underline{A}} = ad - bc \tag{1}$$

provided $\underline{\underline{A}}$ is regular, i.e., the determinant $ad - bc \neq 0$.

(b) (Exercise at home): Show by evaluating the matrix product $\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1}$ that the inverse of regular 3×3 matrices is given by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{aei + bfg + cdh - afh - bdi - ceg} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

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Problem 2.3: Vector and Matrix Derivatives

A vector derivative of a scalar function depending on a vector $\vec{\beta}$ of variables is defined to be the column vector

$$\frac{\partial f(\vec{\beta})}{\partial \vec{\beta}} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial f}{\partial \beta_0} \\ \frac{\partial f}{\partial \beta_1} \\ \vdots \\ \frac{\partial f}{\partial \beta_J} \end{pmatrix}.$$

Apply this definition to the scalar functions $f_1(\vec{\beta}) = \vec{\beta}'\vec{a}$ and $f_2(\vec{\beta}) = \vec{\beta}'\underline{\underline{A}}\vec{\beta}$ ($\vec{a}$ and $\underline{\underline{A}}$ do not depend on $\vec{\beta}$) and show that following derivation rules are valid:

$$\frac{\partial}{\partial \vec{\beta}} \left(\vec{\beta}'\vec{a}\right) = \frac{\partial}{\partial \vec{\beta}} \left(\vec{a}'\vec{\beta}\right) = \vec{a},$$

and

$$\frac{\partial}{\partial \vec{\beta}} \left(\vec{\beta}'\underline{\underline{A}}\vec{\beta}\right) = \left(\underline{\underline{A}} + \underline{\underline{A}}'\right)\vec{\beta}.$$