Methods in Transportation Econometrics and Statistics (Master)

Winter semester 2020/21, Tutorial No. 1a

Problem 1a.1: SI model

Consider the SI infection spread model for the fraction I of infected,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I(1-I) \tag{1}$$

(a) Show that the function

$$I(t) = \frac{\exp(\beta t)}{1+\exp(\beta t)}$$

is a solution to (1) for the initial infection percentage $I(0) = 0.5$

(b) Consider the begin of the infection process with $I << 1$, $S = 1 - I \approx 1$. Show that, in this case, we revert to the model $\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I$ of unlimited growth with the solution $I = I_0 e^{\beta t}$

Problem 1a.2: SIR model

Consider the SIR infection spread model:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta IS,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= +\beta IS - \gamma I,\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= +\gamma I\end{aligned} \tag{2}$$

(a) Show that $S + I + R = \text{const.}$ and can be set $=1$

(b) Argue why the third equation for R is not really needed and can be calculated afterwards once one has a solution for the functions $S(t)$ and $I(t)$. Use arguments from the econometric concept of model linking

(c) Consider the begin of the infection spread where $I \ll 1$, $S = 1 - I \approx 1$ and $R = 0$. Show that, in this case, the middle equation just reverts to a model for unlimited growth with the growth-rate parameter $\beta - \gamma$

(d) Show that *any* solution $S(t) = S_0 = \text{const.} \in [0, 1]$, $I = 0$ and $R(t) = R_0 = 1 - S_0$ is a steady-state solution to (2). Also discuss that any of these solutions represents a situation "pandemic over".

(e) Assume a situation as in (d), i.e., the virus is gone but now it is re-introduced at time $t = 0$ from elsewhere. This means a tiny fraction $\tilde{I}$ of new infections, and we have at time $t = 0$ the situation

$$S(0) = S_0 - \tilde{I}(0), \quad I(0) = \tilde{I}(0), \quad R(0) = R_0$$

and generally $S = S_0 - \tilde{S}$, $I = \tilde{I}$, and $R = R_0 + \tilde{R}$. Show under which conditions this new infection will die out or trigger a new epidemic wave. Also express this condition innterms of the initial growth rate $R_0 = \beta/\gamma$

hint: Insert this into the middle equation of the SIR model (2) and ignore all products of quantities with a tilde since they are much smaller than terms containing only a single quantity with a tilde

Problem 1a.3: Tests: sensitivity and specificity

Notice: This problem is not covid-19 specific but one of the most important problems for any kind of real (not statistical) non-perfect tests having a binary outcome "positive" or "negative".

A population with a real fraction $p = 1\,\%$ of infected persons (also called *incidence*) is tested with a test having a sensitivity $1 - \alpha = 98\,\%$ and a *specificity* $1 - \beta = 0.99\,\%$, i.e. from 100 infected people, on average, 98 are correctly tested as "positive" and from 100 non-infected persons, 99 are tested correctly to be "negative", respectively.

(a) Explain the meaning of α and β. Under which null hypothesis, *alpha* and β would be errors of the first and second kind (also called alpha and beta errors)?

(b) Draw a tree diagram of the situation starting with the distinction infected/not infected and then branching into tested positive/negative.

(c) Determine the expected fraction of positive tests and the conditional probabilities that a positively tested person is infected and a negatively tested person is not infected