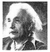
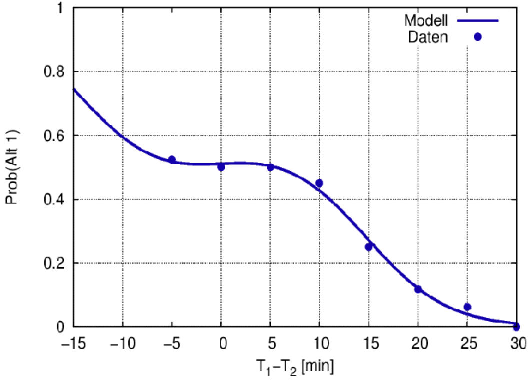
11 Advanced Concepts of Discrete-Choice Theory

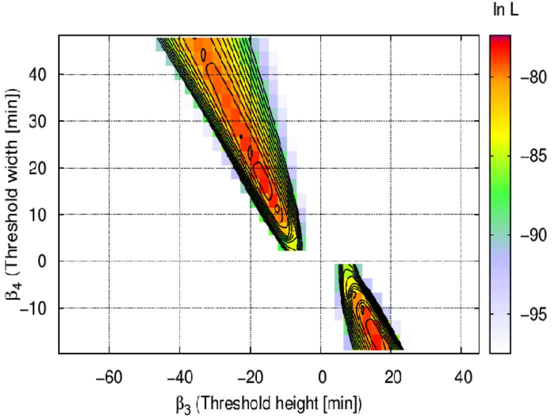
11.1  Parameter Nonlinear Models

Application: Determining subjective thresholds/indifference regions

Person class	Time Alternative 1 [min]	Time Alternative 2 [min]	Choice Alt. 1	Choice Alt. 2
1	25	30	11	10
2	30	30	10	10
3	35	30	10	10
4	40	30	9	11
5	45	30	5	15
6	50	30	2	15
7	55	30	1	15
8	60	30	0	15

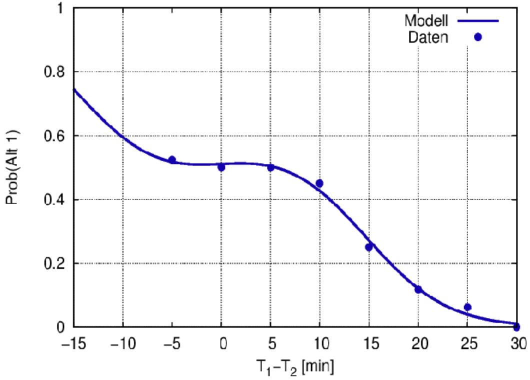
Modelling the threshold

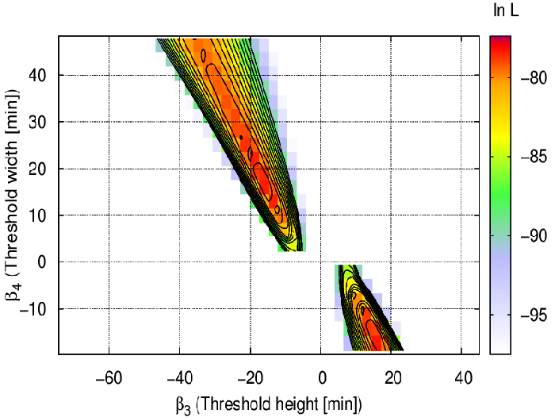




$$V_{n1} - V_{n2} = \beta_1 + \beta_2 \left[\Delta T_n + \beta_3 \tanh\left(\frac{\Delta T_n}{\beta_4} \right) \right]$$

Modelling the threshold





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$\hat{\beta}_1 = 0.043 \pm 0.236$ AC

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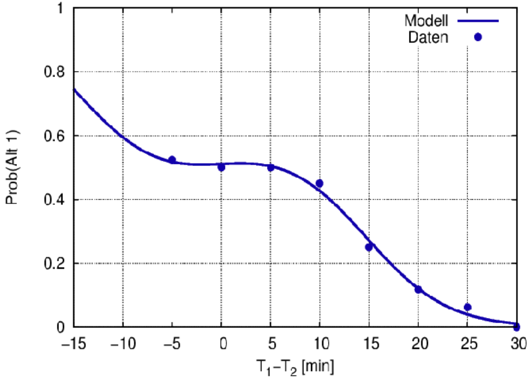

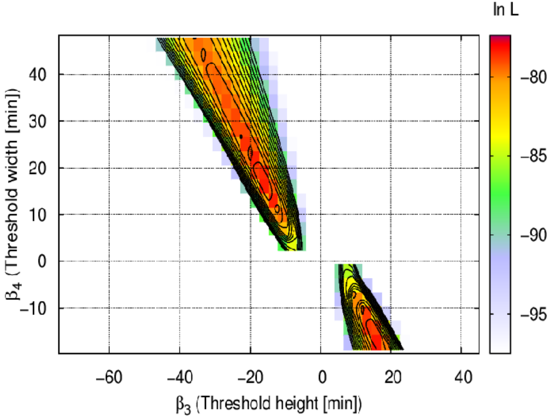

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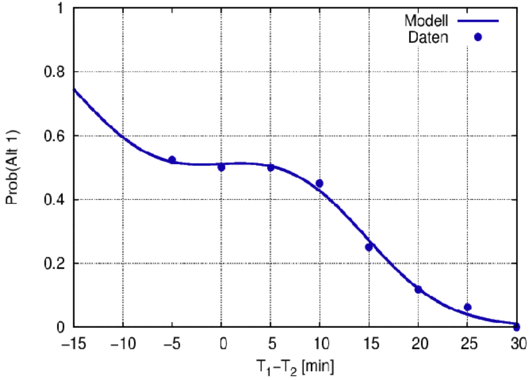

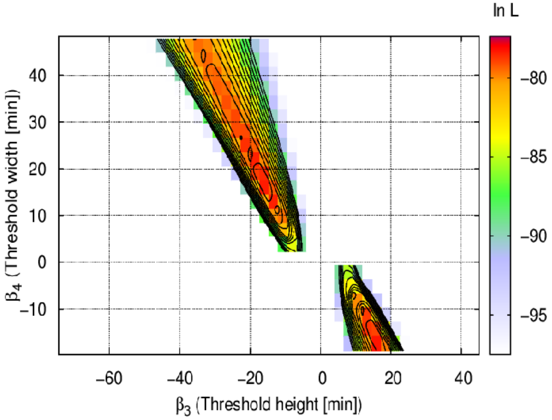

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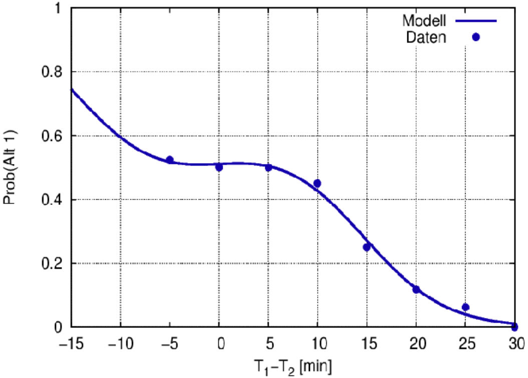

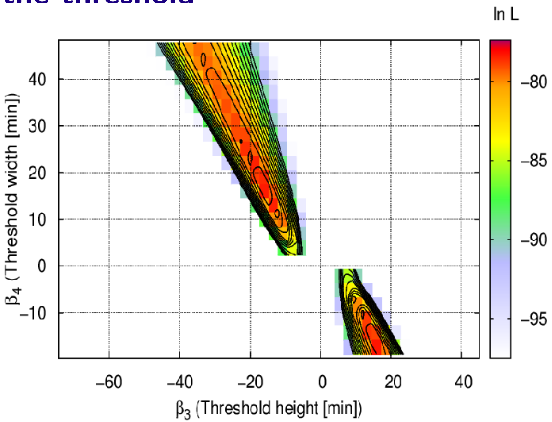

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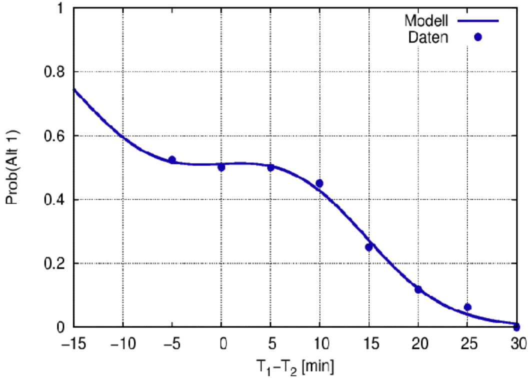

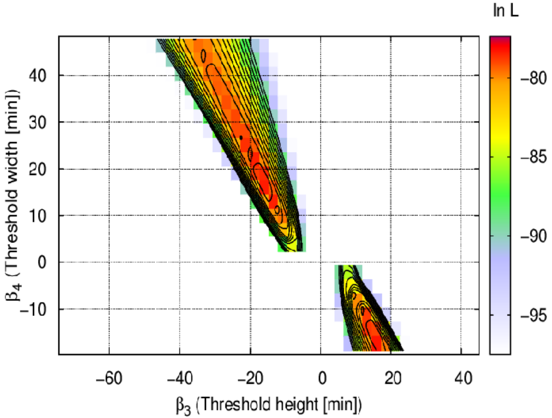

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$\hat{\beta}_3 = -15 \pm 18$ degree of nonlinearity $\geq -\beta_4$

$\hat{\beta}_4 = 14 \pm 21$ threshold width

!! Generally, $L(\boldsymbol{\beta})$ has no longer a unique maximum, here, because of

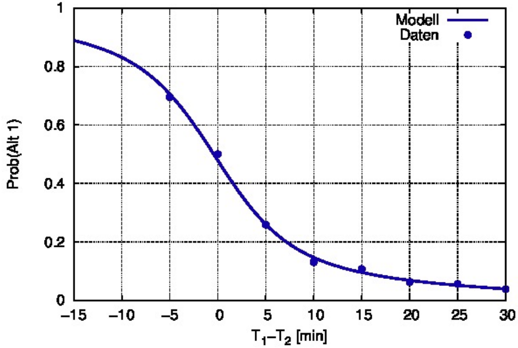
$$\beta_3 \tanh\left(\frac{\Delta T_n}{\beta_4}\right) = -\beta_3 \tanh\left(\frac{\Delta T_n}{-\beta_4}\right)$$

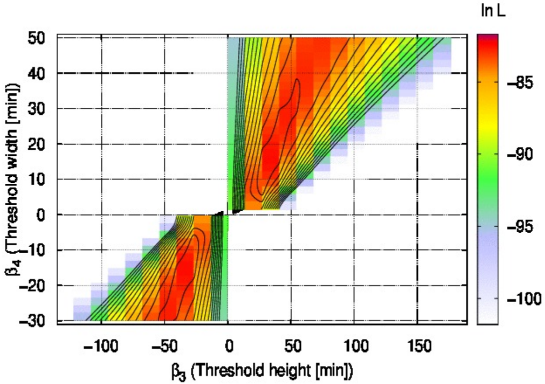
The reverse: Increased sensitivity at reference point

Person class	Time Alternative 1 [min]	Time Alternative 2 [min]	Choice Alt. 1	Choice Alt. 2
1	25	30	16	7
2	30	30	10	10
3	35	30	7	20
4	40	30	3	20
5	45	30	3	25
6	50	30	2	30
7	55	30	1	17
8	60	30	2	50

Such increased sensitivity at the reference (here: equal trip times) is proposed by the **Prospect Theory** of Kahneman/Twersky in certain situations

Modelling the increased sensitivity





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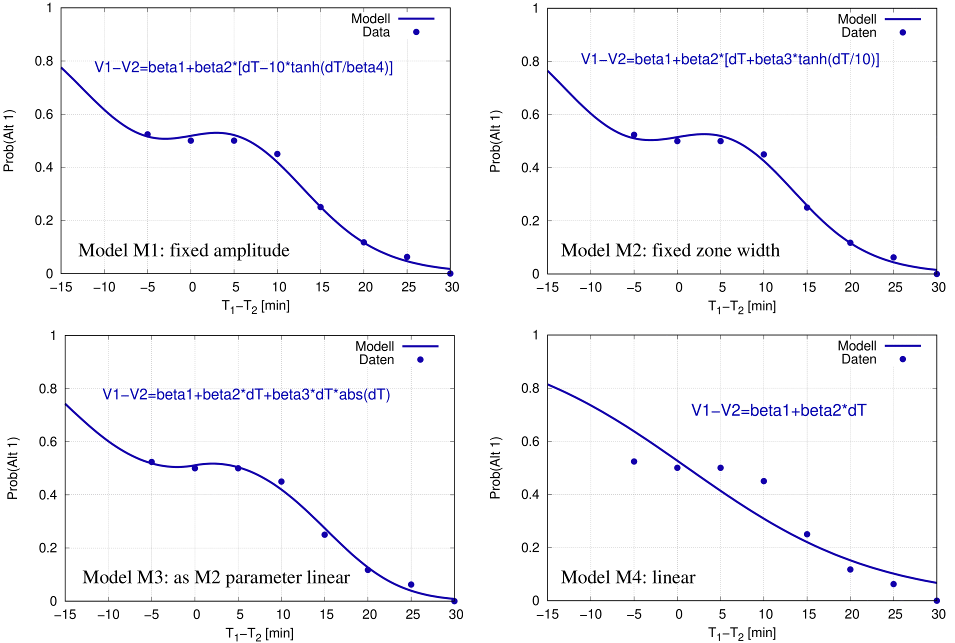
$$\hat{\beta}_1 = -0.08 \pm 0.25,$$
$$\hat{\beta}_2 = -0.05 \pm 0.10,$$
$$\hat{\beta}_3 = 27 \pm 101,$$
$$\hat{\beta}_4 = 10 \pm 16$$

Four further models applied to the threshold data



11.2 GEV and Nested Logit Models

Motivation

When taking decisions, the available options are often coupled in a way that i.i.d. random utilities are not applicable:

- Destination and mode choice
- Destination city and job offers when about to moving
- Expansion of a company: Creating a new branch office and if so, where?

In these cases, a decision involves taking two or more sub-decisions with nearly fixed random utilities in the associated alternative sets, so the total random utility is correlated

$\Rightarrow$ **Red-Bus-Blue-Bus problem.**

$\Rightarrow$ How to model this while retaining explicit expressions for the choice probabilities?

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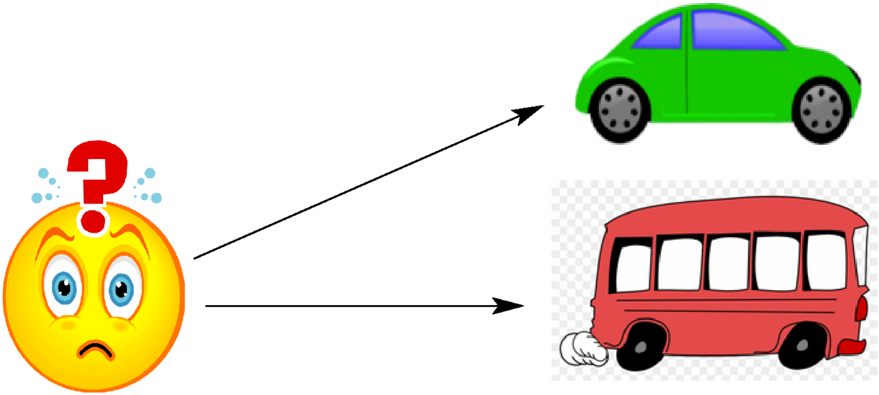
MNL: The Red-Bus-Blue-Bus Problem



Times and costs equal,
AC zero

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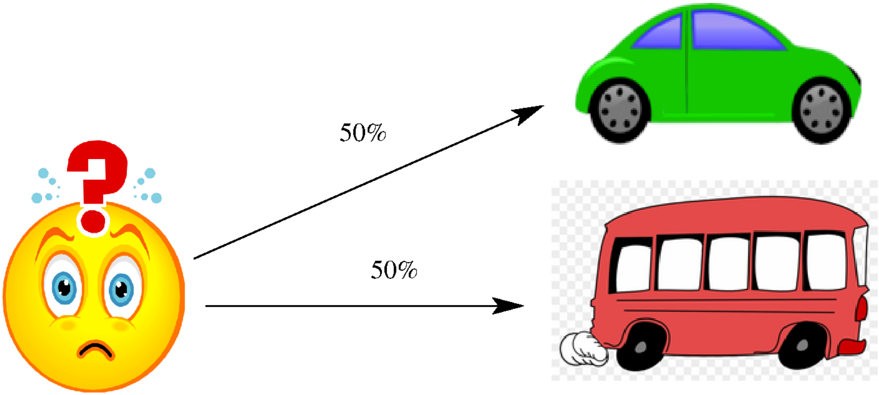

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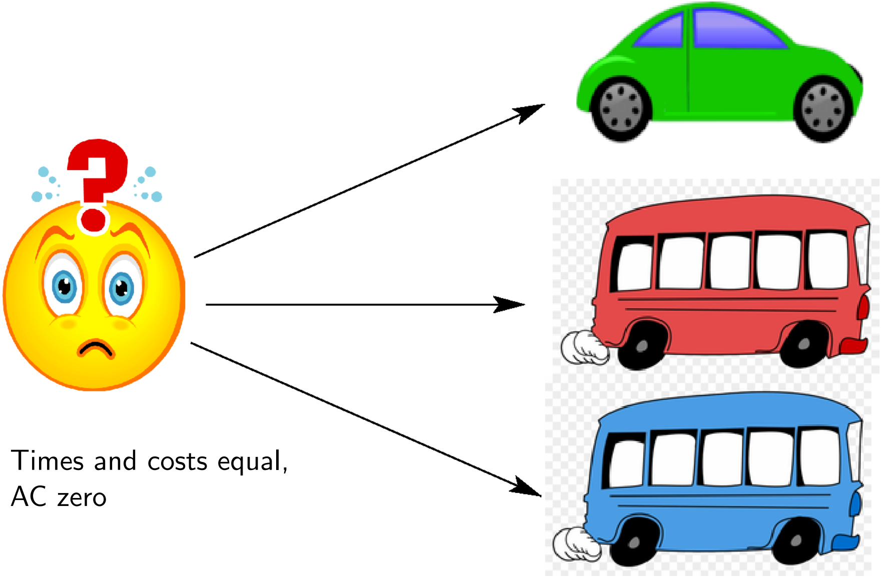

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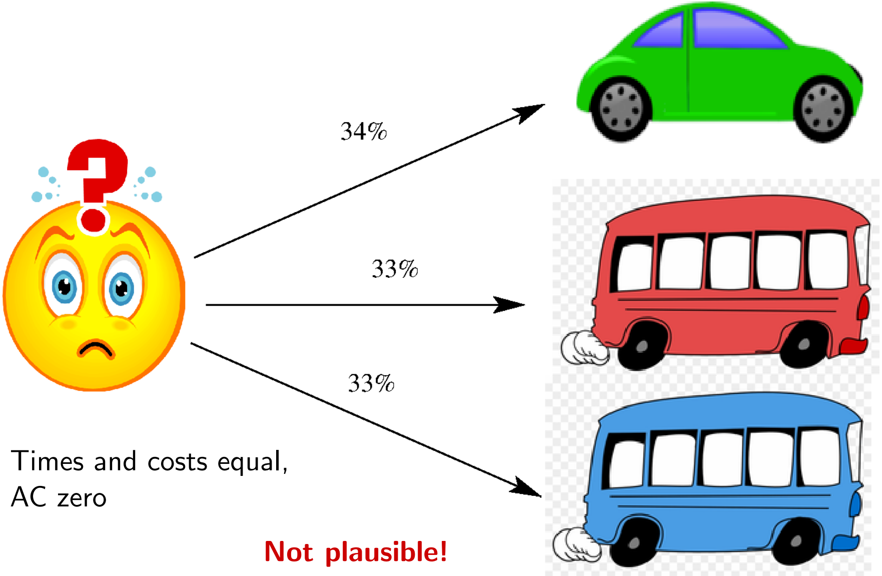
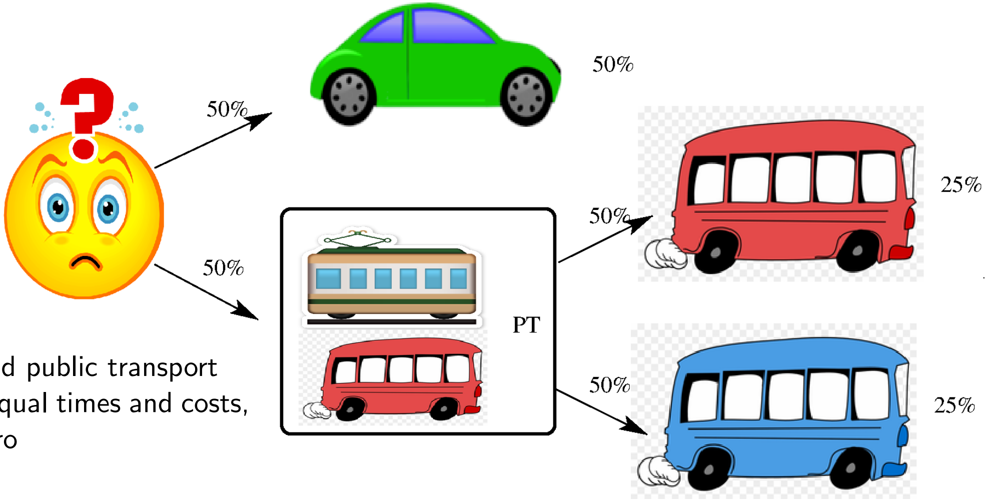


Times and costs equal,
AC zero

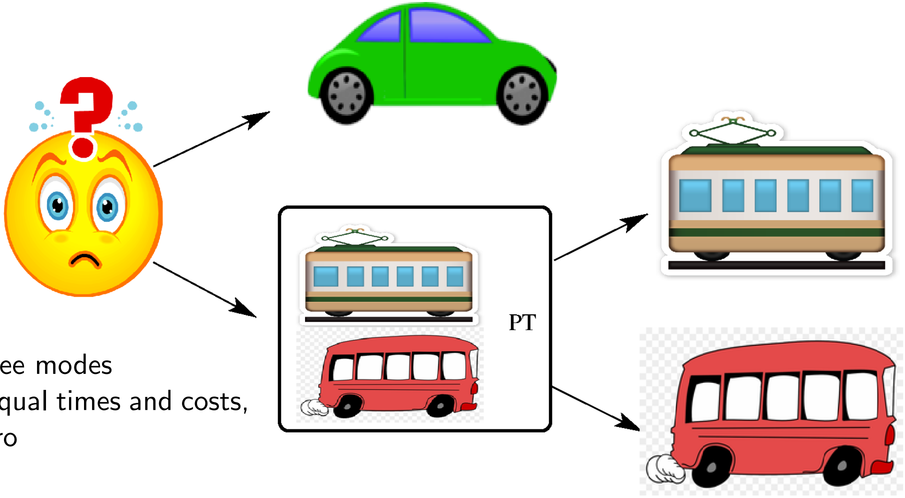
Not plausible!

100% correlated random utilities: Problem solved!



Car and public transport have equal times and costs, AC zero

Nontrivial nested decision: partial correlations



All three modes
have equal times and costs,
AC zero

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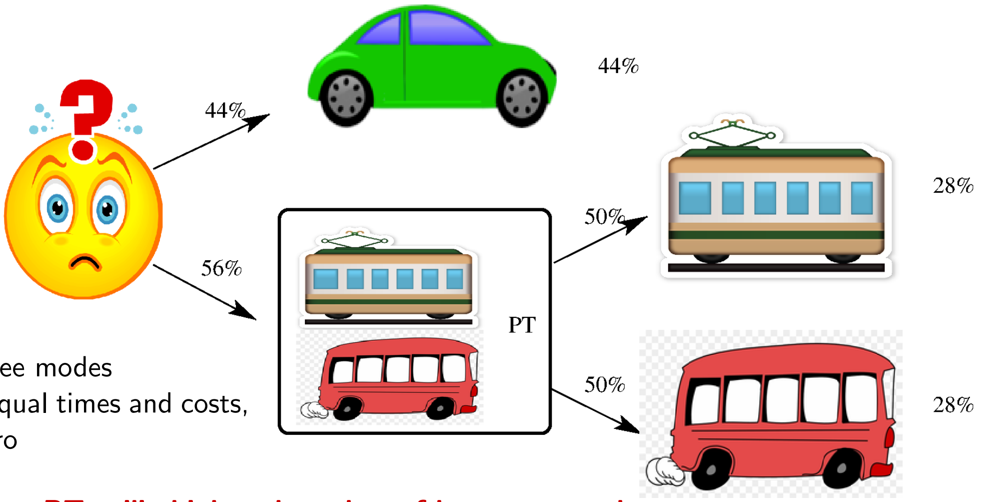

All three modes
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Average PT utiliy higher than that of bus or tram alone because some prefer tram, some bus

The general GEV generating function

All the GEV models are defined via a **Generating function $G(\boldsymbol{y}) = G(y_1, ..., y_I)$** satisfying following formal conditions:

- Not negative: $G(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y}$,
- Asymptotics: $G \to \infty$ if any $y_i \to \infty$,
- Sign of derivatives:

$$G_i \equiv \frac{\partial G}{\partial y_i} \geq 0,$$

$$G_{ij} \equiv \frac{\partial^2 G}{\partial y_i \, \partial y_j} \leq 0 \text{ if } i \neq j,$$

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- Homogeneity of degree 1: $G(\alpha \boldsymbol{y}) = \alpha G(\boldsymbol{y})$

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The Nobel-Price winning result of McFadden et. al.

Any GEV function $G(\boldsymbol{y})$ satisfying the above four conditions

- generates a random vector ϵ satisfying a generalized extreme-value distribution with the distribution function

$$F(\boldsymbol{e}) = P(\epsilon_1 \le e_1, ..., \epsilon_I \le e_I) = e^{-G(\boldsymbol{y})} \text{ with } y_i = e^{-e_i}$$

- has analytic choice probabilities when maximizing the total utilities $U_i = V_i + \epsilon_i$:

$$P_i = \frac{y_i G_i(\boldsymbol{y})}{G(\boldsymbol{y})} \text{ with } G_i = \frac{\partial G}{\partial y_i}, \; y_i = e^{+V_i}$$

? Check why the above conditions for $G(\boldsymbol{y})$ must be true

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Question: Check the conditions for $G(\boldsymbol{y})$

? Why $G(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y}$?

! Because a distribution function $F = e^{-G}$ must be ≤ 1 (the condition $F \geq 0$ is satisfied automatically)

? Why $G \to \infty$ if any $y_i \to \infty$?

! If $y_i \to \infty$ then the argument $e_i = -\ln y_i$ of the distribution function tends to $-\infty$. Since the corresponding random variable ϵ_i is always $> -\infty$, we have $F = e^{-G} = 0$, hence $G \to \infty$

? Sign of derivatives of G?

! We check only the first derivative $G_i = \frac{\partial G}{\partial y_i}$. We have $P_i = y_i G_i / G$ with P_i, $y_i = e^{-e_i}$ and G because of the first requirement all ≥ 0. Hence $G_i \geq 0$. The other conditions follow from the non-negativity of the distribution functions

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! We check only the first derivative $G_i = \frac{\partial G}{\partial y_i}$. We have $P_i = y_i G_i / G$ with P_i, $y_i = e^{-e_i}$ and G because of the first requirement all ≥ 0. Hence $G_i \ge 0$. The other conditions follow from the non-negativity of the distribution functions

? Homogeneity $G(\alpha \boldsymbol{y}) = \alpha G(\boldsymbol{y})$ for any $\alpha > 0$?

! Because of $P_i = y_i G_i / G$ and the scaling invariance $P(\epsilon_1 < e_1) = P(\lambda \epsilon_1 < \lambda e_1)$ with $\alpha = e^\lambda$

Question: Check the conditions for $G(\boldsymbol{y})$

? Why $G(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y}$?

! Because a distribution function $F = e^{-G}$ must be ≤ 1 (the condition $F \geq 0$ is satisfied automatically)

? Why $G \to \infty$ if any $y_i \to \infty$?

! If $y_i \to \infty$ then the argument $e_i = -\ln y_i$ of the distribution function tends to $-\infty$. Since the corresponding random variable ϵ_i is always $> -\infty$, we have $F = e^{-G} = 0$, hence $G \to \infty$

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Special Case I: Multinomial-Logit

- Generating function:

$$G(\boldsymbol{y})^{\text{MNL}} = \sum_{j=1}^{I} y_j$$

- Distribution function of the random utilities (RUs):

$$\begin{aligned} F(\boldsymbol{e}) &= \exp\left[-G\left(e^{-e_1}, \ldots\right)\right] = \exp\left(-\sum_j e^{-e_j}\right) \\ &= \prod_j \exp\left(-e^{-e_j}\right) \ \Rightarrow \ \epsilon_i \sim \text{ i.i.d. Gumbel} \end{aligned}$$

- Choice probabilities:

$$\begin{aligned} G_i &= \frac{\partial G}{\partial y_i} = 1, \\ P_i &= \frac{y_i}{\sum_{j=1}^{I} y_j} = \frac{\exp(V_i)}{\sum_{j=1}^{I} \exp(V_j)} \end{aligned}$$

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Special Case II: Two-level Nested Logit model

- Hierarchical decision: $i=(l,m)$, l: top-level alternatives, m second-level alternatives depending on l
- Generating GEV function:

$$G^{\text{NL}}(\boldsymbol{y})=\sum_{l=1}^{L}\left(\sum_{m=1}^{M_l} y_{lm}^{1/\lambda_l}\right)^{\lambda_l}$$

  where $\lambda_l \in [0,1]$ determine the correlations of the RUs in "nest" l:
  - $\lambda_l \to 1$: Limit of MNL, zero correlation $\Rightarrow$ check it!
  - $\lambda_l \to 0$: no RUs inside the nests, correlation=1: **sequential model: blue and red buses**
- Distribution of the RUs:

$$\begin{aligned} F(\boldsymbol{e}) &= \exp\left[-\sum_l\left(\sum_m e^{-e_{lm}/\lambda_l}\right)^{\lambda_l}\right]=\prod_l \exp\left[-\left(\sum_m e^{-e_{lm}/\lambda_l}\right)^{\lambda_l}\right] \\ &= \prod_l F_l(\boldsymbol{e}_l) \Rightarrow \text{independent at top level} \end{aligned}$$

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  $\Rightarrow$ independent at top level

Nested Logit choice probabilities

Insert $G^{\mathsf{NL}}(\boldsymbol{y})$ into the general expression $P_i = y_i G_i / G$:

$$P_i = P_{lm} = P_l P_{m|l} = \frac{e^{V_{lm}/\lambda_l} \left(\sum_{m'} e^{V_{lm'}/\lambda_l}\right)^{\lambda_l - 1}}{\sum_{l'} \left(\sum_{m'} e^{V_{l'm'}/\lambda_{l'}}\right)^{\lambda_{l'}}}$$

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- Set/assume $\boxed{V_{lm} = W_l + \tilde{V}_{lm}}$
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? Argue that the outer nest decision is a normal MNL with the *effective nest utilities* given by $\lambda_l I_l$. Because for these assumptions P_l has the normal MNL form

? Show that $\lambda_l I_l$ is at least as high as the utility $\tilde{V}_{lm_l^*}$ of the best alternative within the nest and that $\lambda_l I_l = \tilde{V}_{lm_l^*}$ for $\lambda_l \to 0$. All contributions of the sum inside the log are exponentials and thus positive. Furthermore, the ln function is strictly monotonously increasing. Hence, $\lambda_l I_l$ is larger than any single $\tilde{V}_{lm}$ including the maximum. For $\lambda_l \to 0$, only the maximum contributes to the sum

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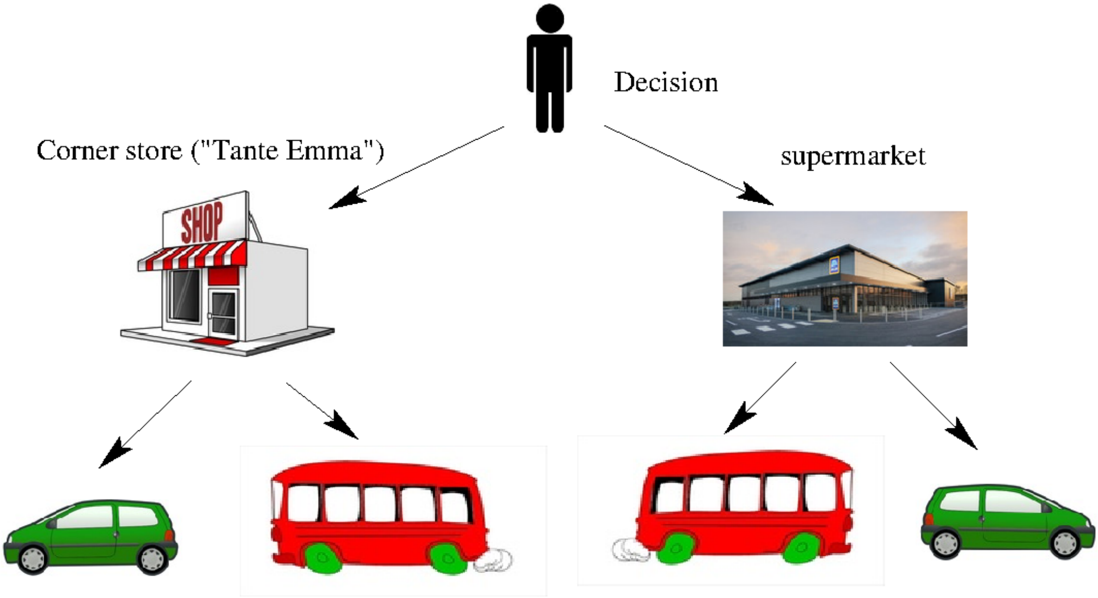
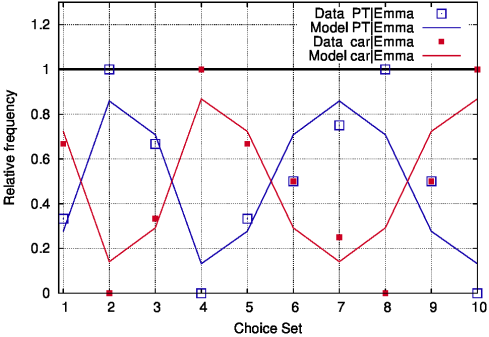
11.2.3 Example: Combined Destination and Mode Choice



Combined destination and mode choice: the data

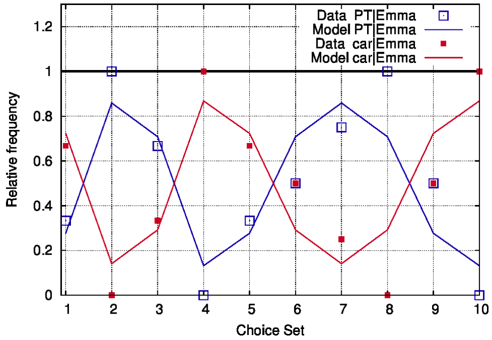
Person group	T [min] Emma, PT	T [min] Emma, car	T [min] superm, PT	T [min] superm, car	Fridge fill level F	y_{11}	y_{12}	y_{21}	y_{22}
1	25	15	25	20	0.9	1	2	0	0
2	25	30	40	30	0.8	3	0	0	1
3	20	20	30	30	0.7	2	1	1	1
4	25	10	25	10	0.6	0	3	0	2
5	15	5	30	20	0.5	1	2	0	2
6	15	15	25	20	0.4	1	1	0	1
7	15	20	45	45	0.3	3	1	0	1
8	15	15	15	15	0.2	1	0	2	3
9	25	15	40	30	0.1	1	1	0	1
10	25	10	25	20	0.0	0	1	1	3

Conditional modal splits



Observed and modelled modal split when driving to "Aunt Emma"

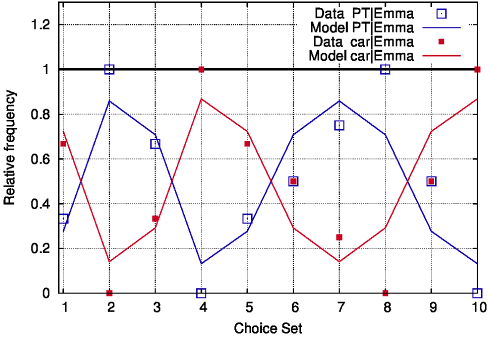
Conditional modal splits



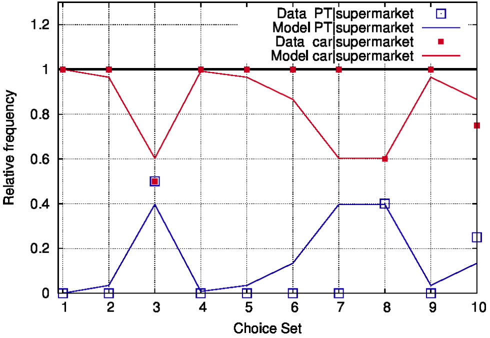
Observed and modelled modal split when driving to "Aunt Emma"

$$
\begin{aligned}
P_{m|n1} &= \frac{\exp(\tilde{V}_{n1m}/\lambda_1)}{\sum_{m'} \exp(\tilde{V}_{n1m'}/\lambda_1)}, \\
\tilde{V}_{n1m}/\lambda_1 &= \beta_1 T_{n1m} + \beta_2 \delta_{m1}, \\
\hat{\beta}_1 &= -0.18, \ \hat{\beta}_2 = +0.88
\end{aligned}
$$

Conditional modal splits



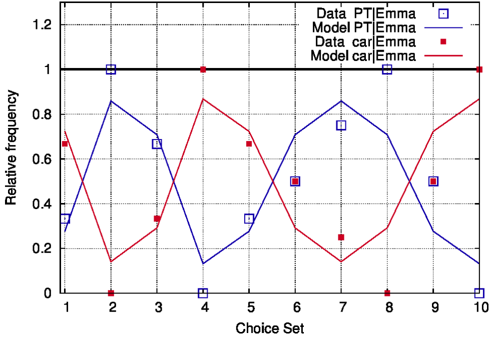
Observed and modelled modal split when driving to "Aunt Emma"



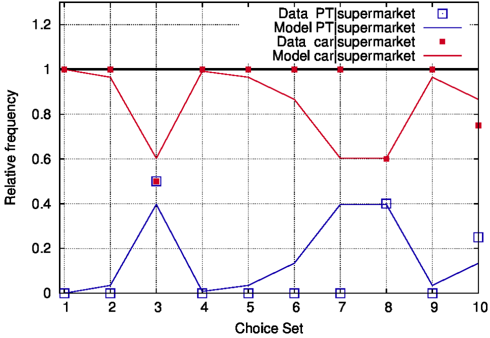
Observed and modelled modal split when driving to the supermarket

$$
\begin{aligned}
P_{m|n1} &= \frac{\exp(\tilde{V}_{n1m}/\lambda_1)}{\sum_{m'}\exp(\tilde{V}_{n1m'}/\lambda_1)}, \\
\tilde{V}_{n1m}/\lambda_1 &= \beta_1 T_{n1m} + \beta_2 \delta_{m1}, \\
\hat{\beta}_1 &= -0.18,\ \hat{\beta}_2 = +0.88
\end{aligned}
$$

Conditional modal splits



Observed and modelled modal split when driving to "Aunt Emma"

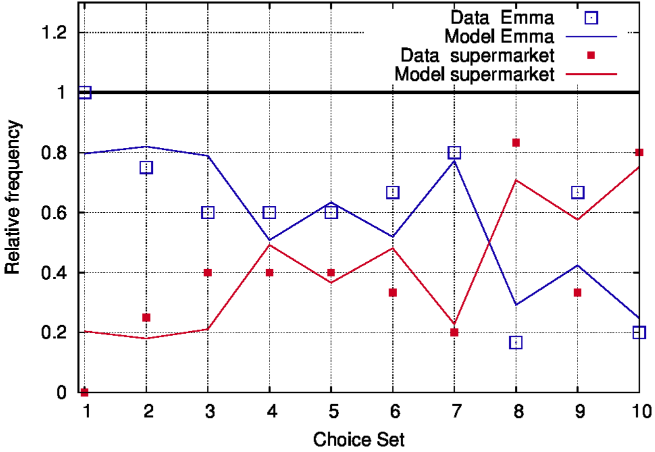


Observed and modelled modal split when driving to the supermarket

$$
\begin{aligned}
P_{m|n1} &= \frac{\exp(\tilde{V}_{n1m}/\lambda_1)}{\sum_{m'} \exp(\tilde{V}_{n1m'}/\lambda_1)}, \\
\tilde{V}_{n1m}/\lambda_1 &= \beta_1 T_{n1m} + \beta_2 \delta_{m1}, \\
\hat{\beta}_1 &= -0.18, \ \hat{\beta}_2 = +0.88
\end{aligned}
$$

$$
\begin{aligned}
P_{m|n2} &= \frac{\exp(\tilde{V}_{n2m}/\lambda_2)}{\sum_{m'} \exp(\tilde{V}_{n2m'}/\lambda_2)}, \\
\tilde{V}_{n2m}/\lambda_2 &= \beta_3 T_{n2m} + \beta_4 \delta_{m1}, \\
\hat{\beta}_3 &= -0.29, \ \hat{\beta}_4 = -0.42
\end{aligned}
$$

Top-level choice of the type of shop



Choice of the type of shop: "Aunt Emma" vs supermarket:

$$P_{nl} = \frac{\exp(W_{nl} + \lambda_l I_{nl})}{\sum_{l'} \exp(W_{nl'} + \lambda_l' I_{nl'})}$$

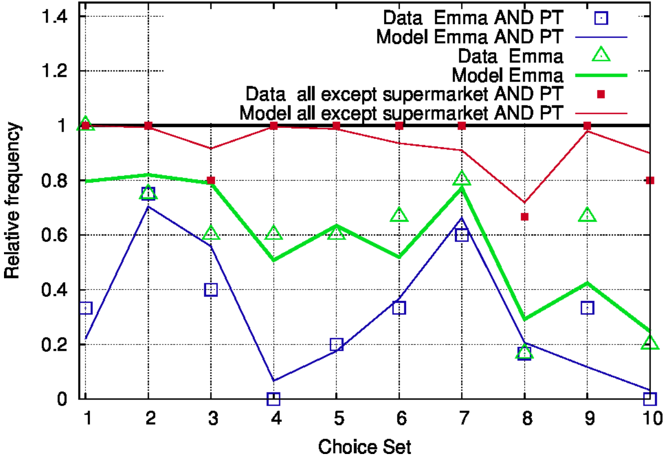
$$W_{nl} = \beta_5 F_n \delta_{l1} + \beta_6 \delta_{l1}$$

$$I_{n1} = \ln \left[\sum_m \exp\left(\hat{\beta}_1 T_{n1m} + \hat{\beta}_2 \delta_{m1} \right) \right]$$

$$I_{n2} = \ln \left[\sum_m \exp\left(\hat{\beta}_3 T_{n2m} + \hat{\beta}_4 \delta_{m1} \right) \right]$$

$$\hat{\beta}_5 = 2.9,\ \hat{\beta}_6 = -2.0,\ \hat{\lambda}_1 = 0.17,\ \hat{\lambda}_2 = 0.21.$$

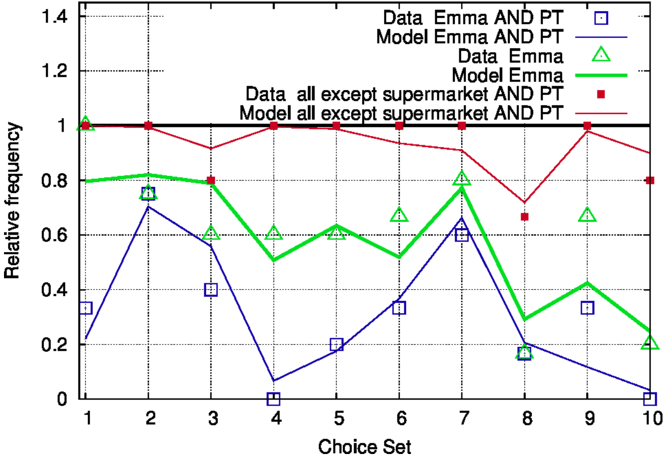
Final combined probabilities



Combined nested choice of shop type and transport mode

$$
\begin{aligned}
P_{ni} &= P_{nl} P_{m|nl} \\
&= \text{Prob(destination)*Prob(mode|destination)}
\end{aligned}
$$

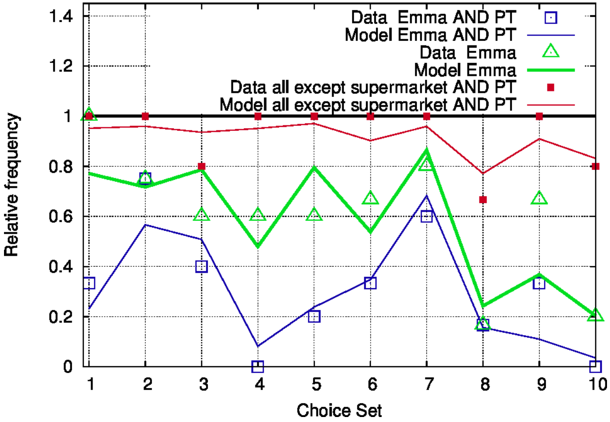
Final combined probabilities



Combined nested choice of shop type and transport mode

$$
\begin{aligned}
P_{ni} &= P_{nl} P_{m|nl} \\
&= \text{Prob(destination)*Prob(mode}|\text{destination)}
\end{aligned}
$$

Counter check: normal MNL



$$P_{ni} = \frac{\exp(V_{ni})}{\sum_{i'=1}^{4} \exp(V_{ni'})}$$

$$\begin{aligned}
V_1 &= \beta_1 T_1 + \beta_2 + \beta_5 F + \beta_6 & (l,m) = (1,1) \text{ Emma+PT} \\
V_2 &= \beta_1 T_2 + \beta_6 + \beta_5 F & (l,m) = (1,2) \text{ Emma+car} \\
V_3 &= \beta_3 T_3 + \beta_4 & (l,m) = (2,1) \text{ supermarket+PT} \\
V_4 &= \beta_3 T_4 & (l,m) = (2,2) \text{ supermarket+car}
\end{aligned}$$

$$\hat{\beta}_1 = -0.15,\ \hat{\beta}_2 = 0.60,\ \hat{\beta}_3 = -0.09,\ \hat{\beta}_4 = -0.84,\ \hat{\beta}_5 = 3.49,\ \hat{\beta}_6 = -1.76$$

11.3 Advanced I: Mixed-Logit Models

if time allows, see German script, Sec. 4.14